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Hydra Battles and AC Termination

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Outline

1. **Battle of Hercules and Hydra**
2. **Termination**
3. **Hydras modulo AC**
4. **Termination modulo AC**
5. **Conclusion**

Battle of Hercules and Hydra (Kirby and Paris 1982)

The mythological monster Hydra is a dragon-like creature with multiple heads. Whenever Hercules in his fight chops off a head, more and more new heads can grow instead, since the beast gets increasingly angry. Hydra dies and Hercules wins if there are no heads left.

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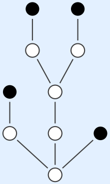

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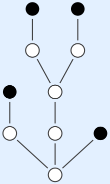

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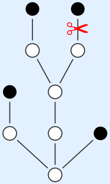

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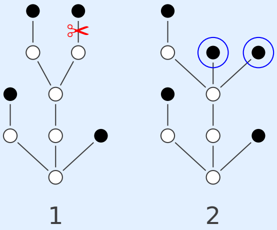

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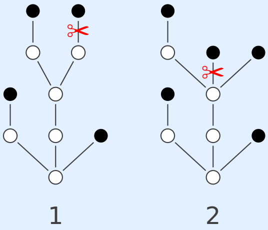

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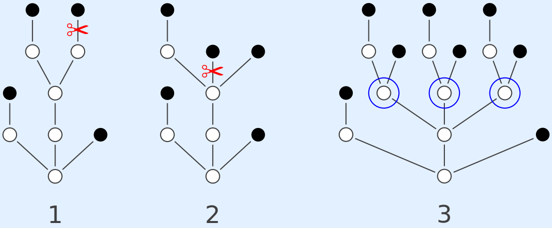

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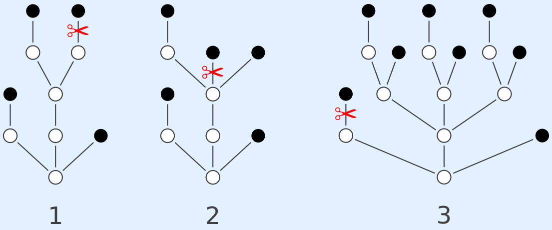

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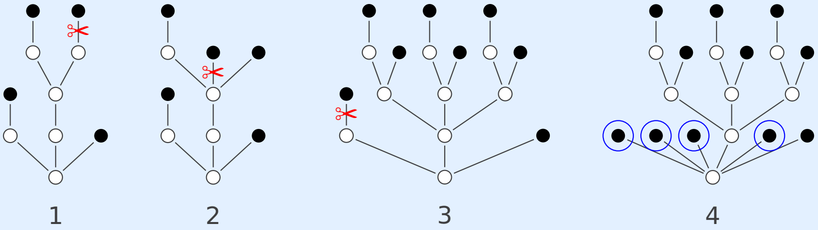

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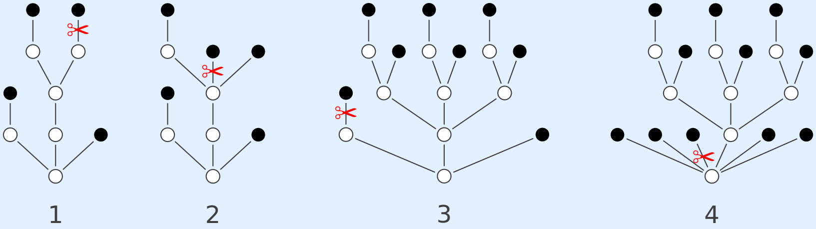

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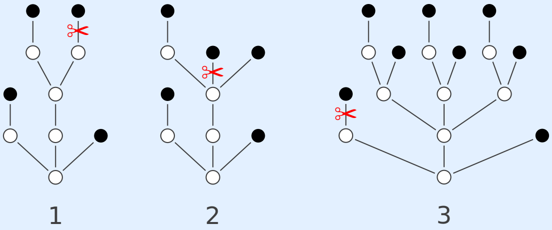

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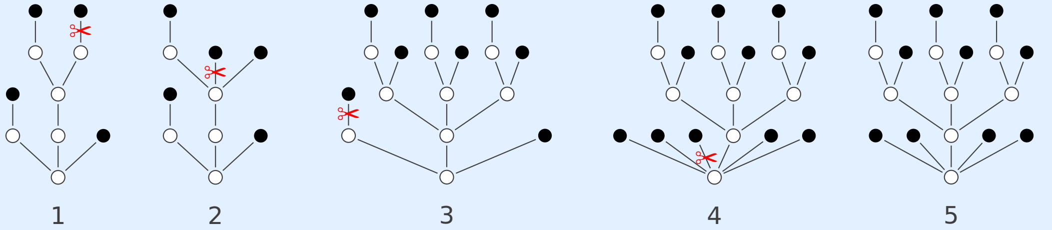

Can Hercules win the battle?

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Definition (Touzet 1998)

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- signature $\quad 0 \quad$ (constant) $\quad \bullet \quad \square \quad \circ \quad$ (unary) $\quad c^1 \quad H \quad$ (binary) $\quad c^2 \quad$ (ternary)
- rewrite rules

$$
\begin{aligned}
\square \circ x &\to \circ \square\, x & H(0,x) &\to \circ\, x \\
\bullet \square\, x &\to \square \bullet \bullet\, x & \bullet\, H(H(0,y),z) &\to c^1(y,z) \\
\circ\, x &\to \bullet \square\, x & \bullet\, H(H(H(0,x),y),z) &\to c^2(x,y,z) \\
\bullet\, x &\to x & \bullet\, c^1(x,y) &\to c^1(x,H(x,y)) \\
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TRS $\mathbb{T}$ models specific strategy for Hercules to battle Hydras up to height 4

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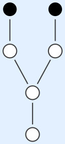

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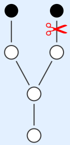

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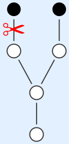

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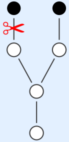

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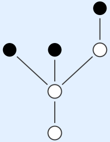

2

$\bullet$ ▯ $\mathsf{H}(\mathsf{H}(\mathsf{H}(0,0),\mathsf{H}(\mathsf{H}(0,0),0)),0) \quad \to^{+} \quad$ $\bullet$ ▯▯ $\mathsf{H}(\mathsf{H}(0,\mathsf{H}(0,\mathsf{H}(\mathsf{H}(0,0),0)),0))$

Outline

Definitions

- well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consists of non-empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) > f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

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Theorem (Lankford 1979; Zantema 1994)

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >_\mathcal{A}$ for well-founded monotone algebra $(\mathcal{A}, >)$

Definition

algebra $(\mathcal{A}, >)$ is simple monotone if every interpretation function $f_{\mathcal{A}}$ is

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Remark

$>_{\mathcal{A}}$ need not be reduction order for well-founded simple monotone algebra $(\mathcal{A}, >)$

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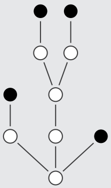

H_1

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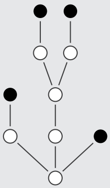
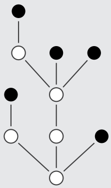


H_1



H_2

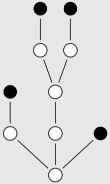
i(i(h) | i(i(i(h) | i(h))) | h) i(i(h) | i(i(i(h) | h | h)) | h)

Remark

to represent Hydras we use h (constant) i (unary) $\mid$ (binary, infix, AC)

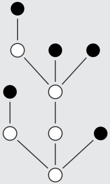
$$x \mid y = y \mid x \qquad\qquad (x \mid y) \mid z = x \mid (y \mid z)$$

Example



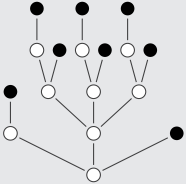
H_1

$\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{h})$



H_2

$\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h} \mid \mathsf{h})) \mid \mathsf{h})$



H_3

$\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}), \mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h}), \mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h}), \mathsf{h})) \mid \mathsf{h})$

$$\mathcal{H} = \mathcal{T}(\{\mathsf{h}, \mathsf{i}, |\})$$

Definition

TRS $\mathbb{H}$

- signature h i |

$\mathcal{H} = \mathcal{T}(\{\mathsf{h}, \mathsf{i}, |\}) \qquad \mathcal{N} = \mathcal{T}(\{\mathsf{0}, \mathsf{s}\})$

Definition

TRS $\mathbb{H}$

- signature h i | 0 (constant) s (unary)

$\mathcal{H} = \mathcal{T}(\{\mathsf{h}, \mathsf{i}, |\}) \qquad \mathcal{N} = \mathcal{T}(\{\mathsf{0}, \mathsf{s}\})$

Definition

TRS $\mathbb{H}$

- signature h i | 0 (constant) s (unary) A B C D (binary) E (unary)

$$\mathcal{H} = \mathcal{T}(\{\mathsf{h}, \mathsf{i}, |\}) \qquad \mathcal{N} = \mathcal{T}(\{0, \mathsf{s}\})$$

Definition

TRS $\mathbb{H}$

- signature h i | 0 (constant) s (unary) A B C D (binary) E (unary)
- rewrite rules

$$\begin{aligned}
\mathsf{A}(n, \mathsf{i}(\mathsf{h})) &\to \mathsf{h} \\
\mathsf{A}(n, \mathsf{i}(\mathsf{h} \,|\, x)) &\to \mathsf{A}(\mathsf{s}(n), \mathsf{i}(x)) \\
\mathsf{A}(n, \mathsf{i}(x)) &\to \mathsf{B}(n, \mathsf{D}(\mathsf{s}(n), \mathsf{i}(x)))
\end{aligned}$$

$$\mathcal{H} = \mathcal{T}(\{\mathsf{h}, \mathsf{i}, |\}) \qquad \mathcal{N} = \mathcal{T}(\{0, \mathsf{s}\})$$

Definition

TRS $\mathbb{H}$

- signature h i | 0 (constant) s (unary) A B C D (binary) E (unary)
- rewrite rules

$$\begin{aligned} \mathsf{A}(n, \mathsf{i}(\mathsf{h})) &\to \mathsf{h} & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x))) &\to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x))) \\ \mathsf{A}(n, \mathsf{i}(\mathsf{h} \mid x)) &\to \mathsf{A}(\mathsf{s}(n), \mathsf{i}(x)) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x) \mid y)) &\to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)) \mid y) \\ \mathsf{A}(n, \mathsf{i}(x)) &\to \mathsf{B}(n, \mathsf{D}(\mathsf{s}(n), \mathsf{i}(x))) \end{aligned}$$

$$\mathcal{H} = \mathcal{T}(\{\mathsf{h}, \mathsf{i}, |\}) \qquad \mathcal{N} = \mathcal{T}(\{0, \mathsf{s}\})$$

Definition

TRS $\mathbb{H}$

- signature h i $|$ $\quad 0$ (constant) s (unary) A B C D (binary) E (unary)
- rewrite rules

$$\begin{aligned}
\mathsf{A}(n, \mathsf{i}(\mathsf{h})) &\to \mathsf{h} \\
\mathsf{A}(n, \mathsf{i}(\mathsf{h} \mid x)) &\to \mathsf{A}(\mathsf{s}(n), \mathsf{i}(x)) \\
\mathsf{A}(n, \mathsf{i}(x)) &\to \mathsf{B}(n, \mathsf{D}(\mathsf{s}(n), \mathsf{i}(x)))
\end{aligned}$$

$$\begin{aligned}
\mathsf{D}(n, \mathsf{i}(\mathsf{i}(x))) &\to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x))) \\
\mathsf{D}(n, \mathsf{i}(\mathsf{i}(x) \mid y)) &\to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)) \mid y) \\
\mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x) \mid y)) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x)) \mid y) \\
\mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x))) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x))) \\
\mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid y)) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{h}) \mid y) \\
\mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}))) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{h}))
\end{aligned}$$

$$\mathcal{H} = \mathcal{T}(\{\mathsf{h}, \mathsf{i}, |\}) \qquad \mathcal{N} = \mathcal{T}(\{0, \mathsf{s}\})$$

Definition

TRS $\mathbb{H}$

- signature h i $|$ $\mathsf{0}$ (constant) s (unary) A B C D (binary) E (unary)
- rewrite rules

$$\begin{aligned}
\mathsf{A}(n, \mathsf{i}(\mathsf{h})) &\to \mathsf{h} & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x))) &\to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x))) \\
\mathsf{A}(n, \mathsf{i}(\mathsf{h} \,|\, x)) &\to \mathsf{A}(\mathsf{s}(n), \mathsf{i}(x)) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x) \,|\, y)) &\to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)) \,|\, y) \\
\mathsf{A}(n, \mathsf{i}(x)) &\to \mathsf{B}(n, \mathsf{D}(\mathsf{s}(n), \mathsf{i}(x))) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \,|\, x) \,|\, y)) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x)) \,|\, y) \\
\mathsf{C}(0, x) &\to \mathsf{E}(x) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \,|\, x))) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x))) \\
\mathsf{C}(\mathsf{s}(n), x) &\to x \,|\, \mathsf{C}(n, x) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}) \,|\, y)) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{h}) \,|\, y) \\
& & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}))) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{h}))
\end{aligned}$$

$$\mathcal{H} = \mathcal{T}(\{h, i, |\}) \qquad \mathcal{N} = \mathcal{T}(\{0, s\})$$

Definition

TRS $\mathbb{H}$

- signature h i | 0 (constant) s (unary) A B C D (binary) E (unary)
- rewrite rules

$$\begin{aligned}
A(n, i(h)) &\to h & D(n, i(i(x))) &\to i(D(n, i(x))) \\
A(n, i(h \mid x)) &\to A(s(n), i(x)) & D(n, i(i(x) \mid y)) &\to i(D(n, i(x)) \mid y) \\
A(n, i(x)) &\to B(n, D(s(n), i(x))) & D(n, i(i(h \mid x) \mid y)) &\to i(C(n, i(x)) \mid y) \\
C(0, x) &\to E(x) & D(n, i(i(h \mid x))) &\to i(C(n, i(x))) \\
C(s(n), x) &\to x \mid C(n, x) & D(n, i(i(h) \mid y)) &\to i(C(n, h) \mid y) \\
i(E(x) \mid y) &\to E(i(x \mid y)) & D(n, i(i(h))) &\to i(C(n, h)) \\
i(E(x)) &\to E(i(x)) & &
\end{aligned}$$

$$\mathcal{H} = \mathcal{T}(\{\mathsf{h}, \mathsf{i}, |\}) \qquad \mathcal{N} = \mathcal{T}(\{0, \mathsf{s}\})$$

Definition

TRS $\mathbb{H}$

- signature h i $|$ 0 (constant) s (unary) A B C D (binary) E (unary)
- rewrite rules

$$\begin{aligned}
\mathsf{A}(n, \mathsf{i}(\mathsf{h})) &\to \mathsf{h} & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x))) &\to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x))) \\
\mathsf{A}(n, \mathsf{i}(\mathsf{h} \mid x)) &\to \mathsf{A}(\mathsf{s}(n), \mathsf{i}(x)) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x) \mid y)) &\to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)) \mid y) \\
\mathsf{A}(n, \mathsf{i}(x)) &\to \mathsf{B}(n, \mathsf{D}(\mathsf{s}(n), \mathsf{i}(x))) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x) \mid y)) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x)) \mid y) \\
\mathsf{C}(0, x) &\to \mathsf{E}(x) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x))) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x))) \\
\mathsf{C}(\mathsf{s}(n), x) &\to x \mid \mathsf{C}(n, x) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid y)) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{h}) \mid y) \\
\mathsf{i}(\mathsf{E}(x) \mid y) &\to \mathsf{E}(\mathsf{i}(x \mid y)) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}))) &\to \mathsf{i}(\mathsf{C}(n, \mathsf{h})) \\
\mathsf{i}(\mathsf{E}(x)) &\to \mathsf{E}(\mathsf{i}(x)) & \mathsf{B}(n, \mathsf{E}(x)) &\to \mathsf{A}(\mathsf{s}(n), x)
\end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ (\to / =_{\mathsf{AC}})^{+} \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ \ (\to / =_{\mathsf{AC}})^+ \ \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\mathsf{A}(0, H_1)$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ (\rightarrow / =_{\mathsf{AC}})^{+} \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\mathsf{A}(0, H_1) \rightarrow \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1))$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ \left(\to / =_{\mathsf{AC}}\right)^{+} \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\mathsf{A}(0, H_1) \to \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \to \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ (\rightarrow / =_{\mathsf{AC}})^{+} \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\begin{aligned}\mathsf{A}(0, H_1) &\rightarrow \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\ &\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))\end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \;\; (\to / =_{\mathsf{AC}})^+ \;\; \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\begin{aligned} \mathsf{A}(0, H_1) &\to \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \to \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\ &\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

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for some $n \in \mathcal{N}$

Example

$$\begin{aligned}
\mathsf{A}(0, H_1) &\rightarrow \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(0, \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))
\end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \;\; (\rightarrow / =_{\mathsf{AC}})^+ \;\; \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\begin{aligned}
\mathsf{A}(0, H_1) &\rightarrow \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))\\
&\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))\\
&\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(0, \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))
\end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ (\rightarrow / =_{\mathsf{AC}})^{+} \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\begin{aligned}
\mathsf{A}(0, H_1) &\rightarrow \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(0, \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
=_{\mathsf{AC}} \cdot &\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))
\end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ (\rightarrow / =_{\mathsf{AC}})^{+} \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\begin{aligned}
\mathsf{A}(0, H_1) &\rightarrow \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(0, \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
=_{\mathsf{AC}} \cdot &\rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \rightarrow \mathsf{B}(0, \mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))
\end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ \ (\to / =_{\mathsf{AC}})^{+} \ \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\begin{aligned}
\mathsf{A}(0, H_1) &\to \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \to \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(0, \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
=_{\mathsf{AC}} \cdot &\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\to \mathsf{B}(0, \mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))
\end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ (\to / =_{\mathsf{AC}})^+ \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\begin{aligned}
\mathsf{A}(0, H_1) &\to \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \to \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(0, \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
=_{\mathsf{AC}} \cdot &\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\to \mathsf{B}(0, \mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{A}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})
\end{aligned}$$

Theorem

if $H, H' \in \mathcal{H} \setminus \{\mathsf{h}\}$ encode successive Hydras in arbitrary battle then

$$\mathsf{A}(n, H) \ (\to / =_{\mathsf{AC}})^{+} \ \mathsf{A}(\mathsf{s}(n), H')$$

for some $n \in \mathcal{N}$

Example

$$\begin{aligned}
\mathsf{A}(0, H_1) &\to \mathsf{B}(0, \mathsf{D}(\mathsf{s}(0), H_1)) =_{\mathsf{AC}} \cdot \to \mathsf{B}(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
&\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(0, \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
=_{\mathsf{AC}} \cdot &\to \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \to \mathsf{B}(0, \mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
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&=_{\mathsf{AC}} \mathsf{A}(\mathsf{s}(0), H_2)
\end{aligned}$$

Outline

Theorem (Middeldorp & Ohsaki 1997)

non-collapsing TRS over many-sorted signature is AC terminating

$\iff$ corresponding TRS over unsorted version of signature is AC terminating

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Definition

two sorts O and N for TRS $\mathbb{H}$ with

$$\mathsf{h} : \mathsf{O} \qquad \mathsf{i}, \mathsf{E} : \mathsf{O} \to \mathsf{O} \qquad | : \mathsf{O} \times \mathsf{O} \to \mathsf{O} \qquad \mathsf{0} : \mathsf{N} \qquad \mathsf{s} : \mathsf{N} \to \mathsf{N} \qquad \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D} : \mathsf{N} \times \mathsf{O} \to \mathsf{O}$$

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Remark

TRS $\mathbb{H}$ is non-collapsing and each rewrite rule consists of well-typed terms of same sort

Definition

$\mathcal{S}$-sorted $\mathcal{F}$-algebra $\mathcal{A} = (\{S_\mathcal{A}\}_{S \in \mathcal{S}}, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ equipped with strict order $>$ on union of all carrier sets is simple monotone if

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Definition

algebra $(\mathcal{A}, >)$ is totally ordered if $>$ is total order

Theorem

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  with $\widehat{f}_{\mathcal{A}}(M_1, \dots, M_n) = \{f_{\mathcal{A}}(m_1, \dots, m_n) \mid (m_1, \dots, m_n) \in M_1 \times \cdots \times M_n\}$

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- $\sqsupset = >_{\mathsf{mul}}$

Proof sketch (cont'd)

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TRS $\mathbb{H}$ is AC terminating

Proof sketch

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- carriers $\mathsf{O}_{\mathcal{A}} = (\mathbb{O} \setminus \{0, 1\}) \times \mathbb{N} \times \mathbb{N}$ and $\mathsf{N}_{\mathcal{A}} = (\mathbb{N} \setminus \{0, 1\}) \times \mathbb{N} \times \mathbb{N}$

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Proof sketch

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- lexicographic order $>_O = (>_{\mathbb{O}}, >, >)_{\text{lex}}$ on $O_{\mathcal{A}}$
- interpretation functions

$$\begin{aligned}
0_{\mathcal{A}} = h_{\mathcal{A}} &= (2,0,0)\\
s_{\mathcal{A}}((n_1,n_2,n_3)) &= (n_1+2,0,0)\\
i_{\mathcal{A}}((x_1,x_2,x_3)) &= (\omega^{x_1}, x_2+1, x_3+1)\\
(x_1,x_2,x_3) \mid_{\mathcal{A}} (y_1,y_2,y_3) &= (x_1 \oplus y_1, x_2+y_2, x_3+y_3)\\
A_{\mathcal{A}}((n_1,n_2,n_3),(x_1,x_2,x_3)) &= (n_1+x_1, n_2+2x_2+2, 0)\\
B_{\mathcal{A}}((n_1,n_2,n_3),(x_1,x_2,x_3)) &= (2+n_1+x_1, n_2+2x_2+1, 0)\\
C_{\mathcal{A}}((n_1,n_2,n_3),(x_1,x_2,x_3)) &= (x_1 \cdot n_1, 0, 0)\\
D_{\mathcal{A}}((n_1,n_2,n_3),(x_1,x_2,x_3)) &= (n_1+x_1, n_2+x_2, n_2+n_3+x_2+x_3)\\
E_{\mathcal{A}}((x_1,x_2,x_3)) &= (x_1, x_2+1, 0)
\end{aligned}$$

Proof sketch

$\{\mathsf{O},\mathsf{N}\}$–sorted algebra $\mathcal{A}$ with

- carriers $\mathsf{O}_\mathcal{A} = (\mathbb{O} \setminus \{0,1\}) \times \mathbb{N} \times \mathbb{N}$ and $\mathsf{N}_\mathcal{A} = (\mathbb{N} \setminus \{0,1\}) \times \mathbb{N} \times \mathbb{N}$
- lexicographic order $>_\mathsf{O} = (>_\mathbb{O}, >, >)_{\mathsf{lex}}$ on $\mathsf{O}_\mathcal{A}$
- interpretation functions

$$\begin{aligned}
\mathsf{0}_\mathcal{A} = \mathsf{h}_\mathcal{A} &= (2,0,0) \\
\mathsf{s}_\mathcal{A}((n_1,n_2,n_3)) &= (n_1+2,0,0) \\
\mathsf{i}_\mathcal{A}((x_1,x_2,x_3)) &= (\omega^{x_1}, x_2+1, x_3+1) \\
(x_1,x_2,x_3) \mid_\mathcal{A} (y_1,y_2,y_3) &= (x_1 \oplus y_1, x_2+y_2, x_3+y_3) \\
\mathsf{A}_\mathcal{A}((n_1,n_2,n_3),(x_1,x_2,x_3)) &= (n_1+x_1, n_2+2x_2+2, 0) \\
\mathsf{B}_\mathcal{A}((n_1,n_2,n_3),(x_1,x_2,x_3)) &= (2+n_1+x_1, n_2+2x_2+1, 0) \\
\mathsf{C}_\mathcal{A}((n_1,n_2,n_3),(x_1,x_2,x_3)) &= (x_1 \cdot n_1, 0, 0) \\
\mathsf{D}_\mathcal{A}((n_1,n_2,n_3),(x_1,x_2,x_3)) &= (n_1+x_1, n_2+x_2, n_2+n_3+x_2+x_3) \\
\mathsf{E}_\mathcal{A}((x_1,x_2,x_3)) &= (x_1, x_2+1, 0)
\end{aligned}$$

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- $|_{\mathcal{A}}$ is strictly monotone

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- $\mathrm{AC} \subseteq {=_{\mathcal{A}}}$

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- $\mathsf{AC} \subseteq =_{\mathcal{A}}$
- all interpretation functions are simple and monotone
- $\mathbb{H} \subseteq >_{\mathcal{A}}$

$$\mathsf{A}(n, \mathsf{i}(\mathsf{h})) \to \mathsf{h}$$

translates to

$$(\omega^2, \text{—}, \text{—}) >_{\mathsf{O}} (2, \text{—}, \text{—})$$

Proof sketch

- $|_{\mathcal{A}}$ is strictly monotone
- AC $\subseteq =_{\mathcal{A}}$
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- $\mathbb{H} \subseteq >_{\mathcal{A}}$

$$\mathsf{A}(n, \mathsf{i}(\mathsf{h} \mid x)) \rightarrow \mathsf{A}(\mathsf{s}(n), \mathsf{i}(x))$$

translates to

$$(\omega^{x_1+2}, \text{—}, \text{—}) >_{\mathsf{o}} (\omega^{x_1}, \text{—}, \text{—})$$

Proof sketch

- $|_{\mathcal{A}}$ is strictly monotone
- $\mathsf{AC} \subseteq =_{\mathcal{A}}$
- all interpretation functions are simple and monotone
- $\mathbb{H} \subseteq >_{\mathcal{A}}$

$$\mathsf{A}(n, \mathsf{i}(x)) \to \mathsf{B}(n, \mathsf{D}(\mathsf{s}(n), \mathsf{i}(x)))$$

translates to

$$(\omega^{x_1}, n_2 + 2x_2 + 4, —) >_{\mathsf{o}} (\omega^{x_1}, n_2 + 2x_2 + 3, —)$$

Proof sketch

- $|_{\mathcal{A}}$ is strictly monotone
- AC $\subseteq =_{\mathcal{A}}$
- all interpretation functions are simple and monotone
- $\mathbb{H} \subseteq >_{\mathcal{A}}$

$$\mathsf{D}(n, \mathsf{i}(\mathsf{i}(x))) \rightarrow \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)))$$

translates to

$$\left(\omega^{\omega^{x_1}},\ n_2+x_2+2,\ n_2+n_3+x_2+x_3+4\right) >_{\mathsf{O}} \left(\omega^{\omega^{x_1}},\ n_2+x_2+2,\ n_2+n_3+x_2+x_3+3\right)$$

Proof sketch

- $|_{\mathcal{A}}$ is strictly monotone
- AC $\subseteq =_{\mathcal{A}}$
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- $\mathbb{H} \subseteq >_{\mathcal{A}}$

$$\mathsf{D}(n, \mathsf{i}(\mathsf{i}(x) \,|\, y)) \to \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)) \,|\, y)$$

translates to

$$\left(\omega^{\omega^{x_1}} \oplus y_1,\ n_2+x_2+y_2+2,\ n_2+n_3+x_2+x_3+y_2+y_3+4\right) >_{\mathrm{O}} \left(\omega^{\omega^{x_1}} \oplus y_1,\ n_2+x_2+y_2+2,\ n_2+n_3+x_2+x_3+y_2+y_3+3\right)$$

Proof sketch

- $|_{\mathcal{A}}$ is strictly monotone
- AC $\subseteq$ $=_{\mathcal{A}}$
- all interpretation functions are simple and monotone
- $\mathbb{H} \subseteq >_{\mathcal{A}}$

$$\mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x) \mid y)) \to \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x)) \mid y)$$

translates to

$$(\omega^{\omega^{x_1+2} \oplus y_1}, —, —) >_{\mathrm{o}} (\omega^{\omega^{x_1} \cdot n_1 \oplus y_1}, —, —)$$

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translates to

$$(\omega^{\omega^2}, -, -) >_{\mathsf{o}} (\omega^{2n_1}, -, -)$$

Proof sketch

- $|_{\mathcal{A}}$ is strictly monotone
- AC $\subseteq =_{\mathcal{A}}$
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$$\mathsf{C}(0, x) \to \mathsf{E}(x)$$

translates to

$$(x_1 \cdot 2, —, —) >_{\mathrm{O}} (x_1, —, —)$$

Proof sketch

- $|_{\mathcal{A}}$ is strictly monotone
- AC $\subseteq =_{\mathcal{A}}$
- all interpretation functions are simple and monotone
- $\mathbb{H} \subseteq >_{\mathcal{A}}$

$$\mathsf{C}(\mathsf{s}(n), x) \;\to\; x \,|\, \mathsf{C}(n, x)$$

translates to

$$(x_1 \cdot (n_1 + 2), \text{—}, \text{—}) \;>_{\mathsf{O}}\; (x_1 \cdot (n_1 + 1), \text{—}, \text{—})$$

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translates to

$$(\omega^{x_1 \oplus y_1}, x_2 + y_2 + 2, y_3 + 1) >_{\mathsf{O}} (\omega^{x_1 \oplus y_1}, x_2 + y_2 + 2, 0)$$

Proof sketch

- $|_{\mathcal{A}}$ is strictly monotone
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$$\mathsf{B}(n, \mathsf{E}(x)) \ \to\ \mathsf{A}(\mathsf{s}(n), x)$$

translates to

$$(2 + n_1 + x_1, n_2 + 2x_2 + 3, \text{—}) \ >_{\mathrm{o}}\ (2 + n_1 + x_1, 2x_2 + 2, \text{—})$$

Outline

1. Battle of Hercules and Hydra
2. Termination
3. Hydras modulo AC
4. Termination modulo AC
5. **Conclusion**

Concluding Remarks

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Postdoc Position

- 3-year project position in Innsbruck
- FWF / JSPS project ARI: Automation of Rewriting Infrastucture
- `https://ari-informatik.uibk.ac.at/position.php`
- application deadline: June 15, 2022