The paint pot problem and common multiples in monoids

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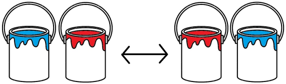
- Swap two consecutive non-empty pots 

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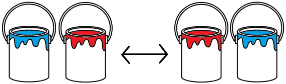
- Swap two consecutive non-empty pots 
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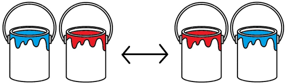
- Swap two consecutive non-empty pots 

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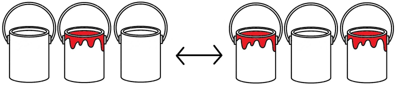
Also reverse allowed: 

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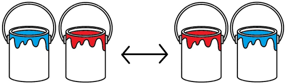
- Swap two consecutive non-empty pots 
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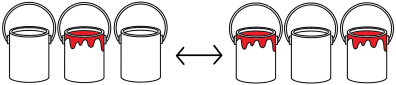
  Also reverse allowed: 

Is it possible to start by a sequence in which the first four pots contain paint in four different colors, and get the first pot empty?

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Question:

Do words x, y exist such that $bcdex =_E ay$?

More general, the set of words = strings over a finite alphabet modulo a set of equations is called a (finitely generated) *monoid*, with concatenation as operation and the empty word as unit element

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So the paint pot problem asks whether a particular monoid satisfies this property for the words *bcde* and *a*

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Unfortunately, all confluence tools fail for proving or disproving confluence for this system for the paint pot problem and variants

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For 3 colors the answer is 'yes'

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$$aba = bab,\ aca = cac,\ ada = dad,\ bc = cb,\ bd = db,\ cd = dc$$

and the smallest solution is

$$\begin{array}{cccccc}
 & bc\underline{dad}cabacda & = & bcad\underline{aca}bacda & = & bcadca\underline{cb}acda \\
= & bcadcab\underline{cac}da & = & bcadcabac\underline{ada} & = & bcadc\underline{aba}cdad \\
= & bca\underline{dc}babcdad & = & b\underline{cac}dbabcdad & = & baca\underline{db}abcdad \\
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Theorem

For words u, v there are no x, y satisfying $ux =_E vy$ if and only if there exists a model M for E such that

$$\{u_M(m) \mid m \in M\} \cap \{v_M(m) \mid m \in M\} = \emptyset$$

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Remark: this proof is very similar to the equivalence of termination and the existence of a monotone algebra that I proved in my very first TERESE talk in January 1991

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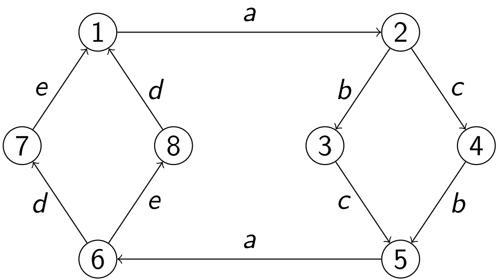

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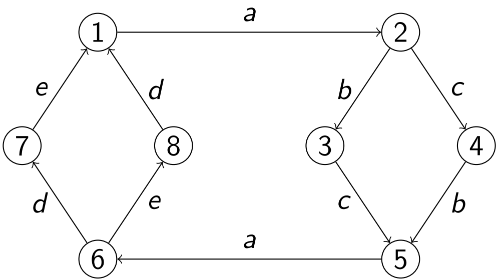

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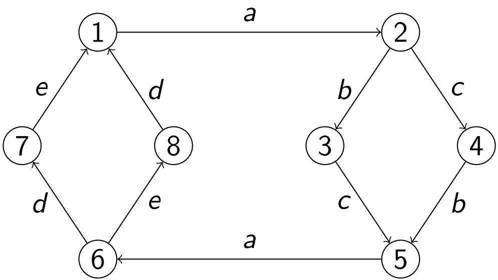

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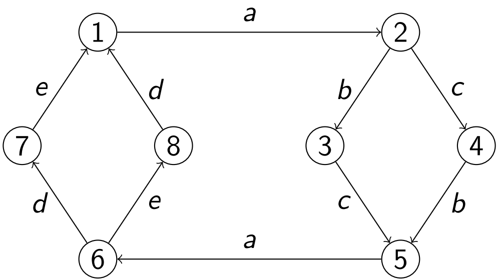

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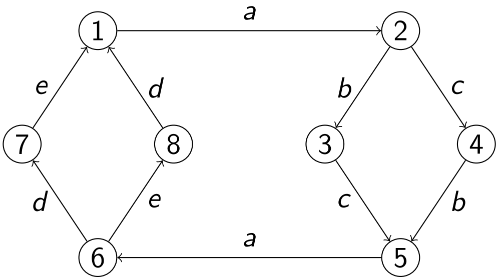

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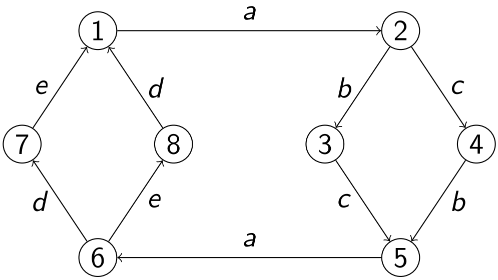

We compute

$$\{b_M(c_M(d_M(e_M(m)))) \mid m \in M\} = \{1, 5\}$$

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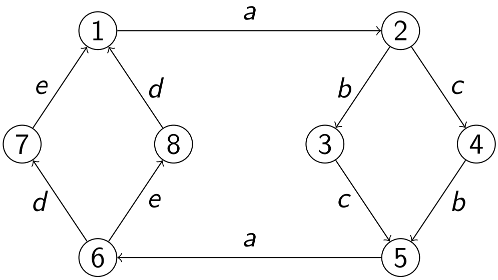

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indeed disjoint

This proof was found by looking for a finite model with n elements for $n = 2, 3, 4, \ldots$, and expressing the requirements in an SMT formula, until for $n = 8$ the formula was satisfiable, and the satisfying assignment yielded the given solution

Generalization to graphs

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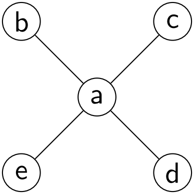
A question posed by Jan Willem Klop is: for which graphs every two words have a common right multiple?

His first conjecture was that this holds iff the graph is acyclic, but this was contradicted by our paint pot problem, since that corresponds to the graph



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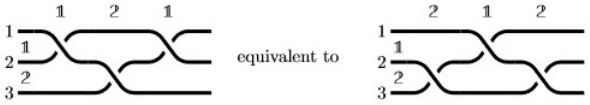

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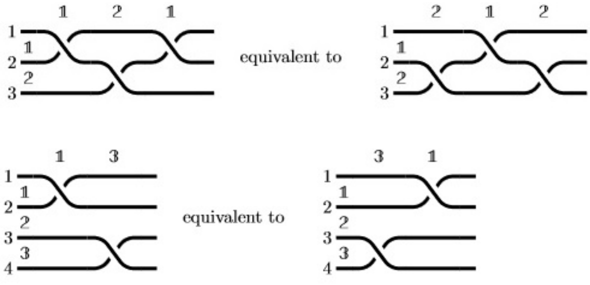

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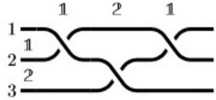

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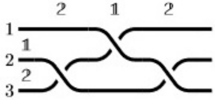

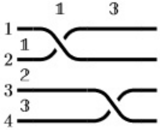

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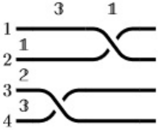

So we get exactly our equations $aba = bab$ for consecutive symbols, and $ab = ba$ for the others

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Positive results, so proving that common right multiples do exist, are obtained by constructing a special word Δ

Our criteria for Δ do not coincide with the standard theory, but are simpler in our view

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Theorem

Let Δ be both init flexible and rotation flexible
Then every u, v have common right multiples, that is, x, y exist with $ux =_E vy$

For many examples we systematically find an init flexible word Δ and check that it is rotation flexible, hence proving that every u, v have common right multiples

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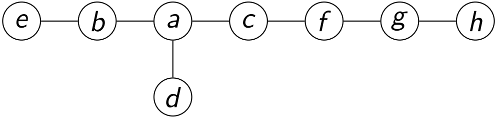

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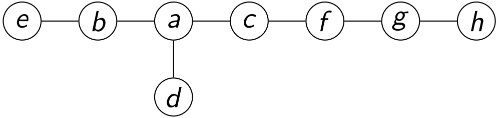

yielding

$\Delta = abacabdabcadebadcabefcabedabcafcdabegfcabedabcafcgfdacbae$

$bdacfghgfcadbacfghebacfgdacfbacdabebadcabfcagfchgfdacbaebdacfgh$

of length 120

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Define $R = R_E$ to consist of the following rewrite rules:

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So common right multiples for u and v are found by rewriting Vu to normal form

Example:

$aba = bab, aca = cac, ada = dad, bc = cb, bd = db, cd = dc$

yields

$$\begin{array}{llllllllllll}
Aa & \to & \epsilon & Ba & \to & abAB & Ca & \to & acAC & Da & \to & adAD \\
Ab & \to & baBA & Bb & \to & \epsilon & Cb & \to & bC & Db & \to & bD \\
Ac & \to & caCA & Bc & \to & cB & Cc & \to & \epsilon & Dc & \to & cD \\
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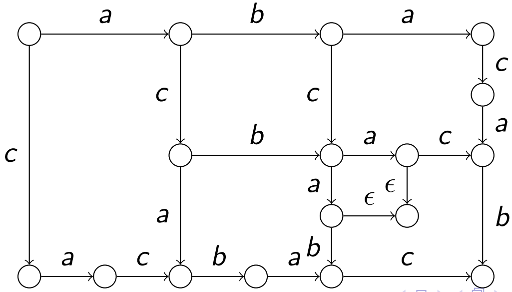

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Unfortunately, one reviewer remarked that our main result was already known: common right multiples in these monoids is equivalent to finiteness of Coxeter groups, which was already fully classified by Coxeter in 1935

Mixed feelings: a pity that the paper was rejected, but scientifically very nice that our main question turned out to be equivalent to a natural question from geometry that seems to be unrelated

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Our approach of disproving common right multiples by finding a model is new, and completely different from existing techniques

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The paint pot problem is a nice and hard puzzle in itself, and its solution is an instance of this new approach