

# Termination by Tiling with PBPO<sup>+</sup> (work in progress)

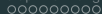
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Roy Overbeek

joint work with Jörg Endrullis

8 June 2022, **TeReSe seminar**

Vrije Universiteit Amsterdam, The Netherlands



# This Talk in a Nutshell

there are many flavours of graphs

⇒ so having an abstract (graph) rewriting mechanism is useful

⇒ PBPO<sup>+</sup> is such a mechanism, and it is expressive



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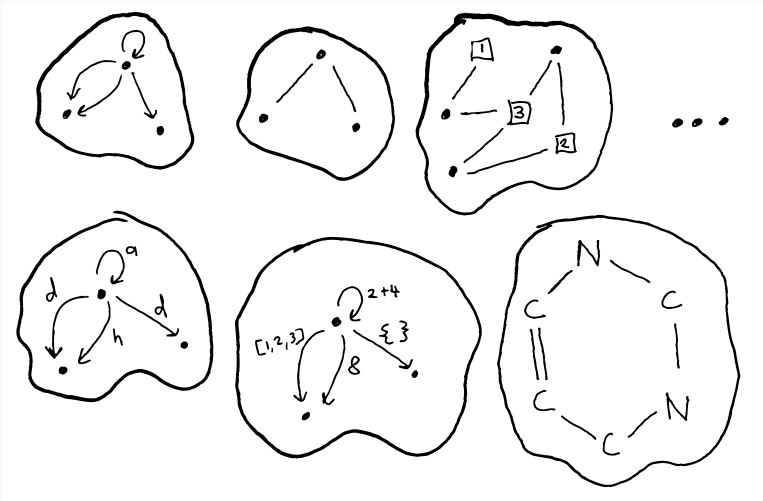
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⇒ PBPO<sup>+</sup> is such a mechanism, and it is expressive

termination is interesting for all flavours of rewriting

⇒ we developed a termination method in an abstract setting for PBPO<sup>+</sup>

# What is a Graph?



## Graph Rewriting: Replacement in Context

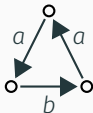
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in a graph, and delete it.”

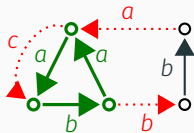
# Graph Rewriting: Replacement in Context

Example rule: “Find an occurrence of



in a graph, and delete it.”

Problem:



What should happen with the red edges?

# Algebraic Graph Transformation

**Algebraic Graph Transformation:** research field since the 70s.

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Example formalisms:

- Double-Pushout (DPO) [Ehrig et al., 1973]
- Single-Pushout (SPO) [Löwe, 1993]
- Sesqui-Pushout (SqPO) [A.Corradini et al., 2006]
- AGREE [Corradini et al., 2015]
- Pullback-Pushout (PBPO) [Corradini et al., 2017]
- **Pullback-Pushout plus Strong Matching (PBPO<sup>+</sup>)** [Overbeek et al., 2021]
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Different frameworks

- use different constructions;
- handle replacement in context differently; and
- make different assumptions about the underlying category.

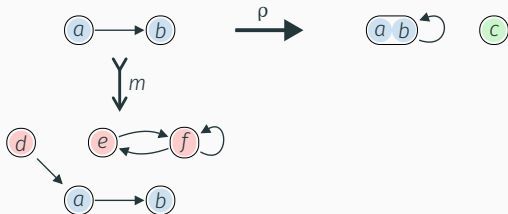
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Rule  $\rho$ : “identify nodes  $a$  and  $b$ , and add a node  $c$ ”:



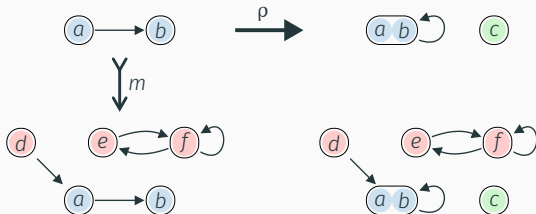
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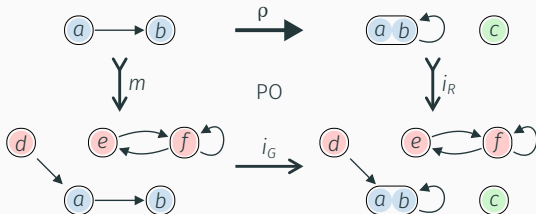
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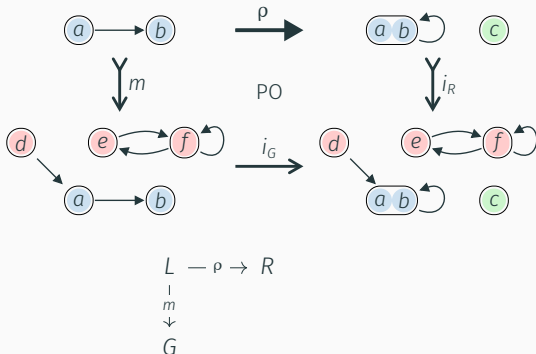
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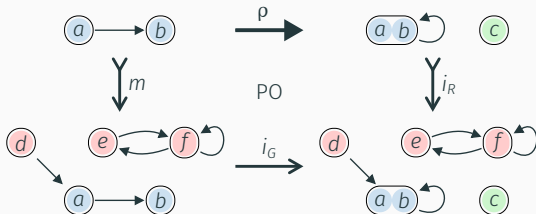
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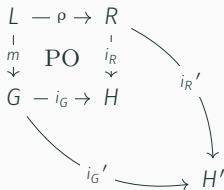
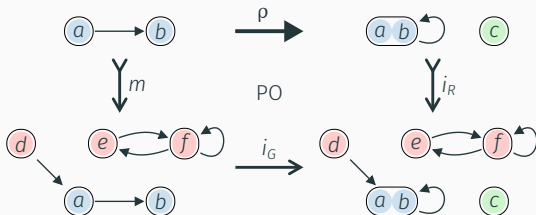
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$$\begin{array}{ccc}
 L & \xrightarrow{\rho} & R \\
 \downarrow m & & \downarrow i_R \\
 G & \xrightarrow{i_G} & H
 \end{array}
 \text{ PO }$$

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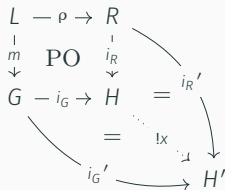
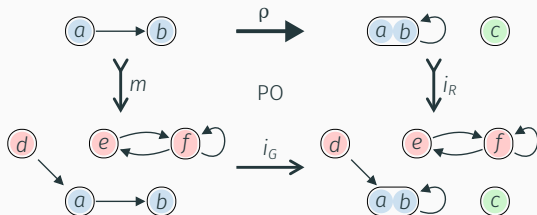
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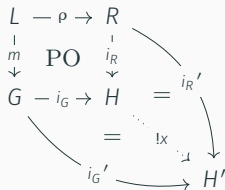
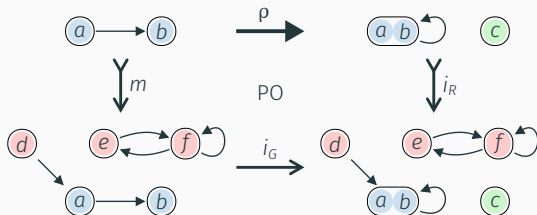
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Think of a pushout as a **gluing construction** or a **fibred union**.

## Construction #2: Pullback

The **dual** of a pushout is a **pullback**:

$$\begin{array}{c} G \\ | \\ \alpha \\ \downarrow \\ L \leftarrow \rho - R \end{array}$$

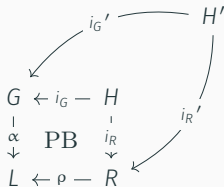
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$$\begin{array}{ccccc}
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 & & & = & \\
 & & & \overset{!x}{\dashrightarrow} & \\
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 \end{array}$$

Think of a pullback as a **fibred product** or as a **generalized intersection**.



# PBPO<sup>+</sup>: Pullback-Pushout plus Strong Matching

Definition (PBPO<sup>+</sup> Rule [Corradini et al., 2017, Overbeek et al., 2021])

“patterns”:

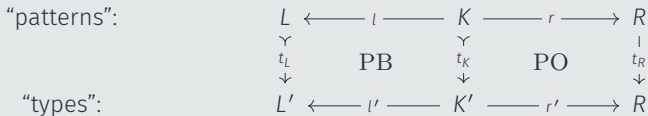
$$\begin{array}{ccccc}
 L & \longleftarrow l & K & \longrightarrow r & R \\
 \Upsilon & & \Upsilon & & | \\
 t_L & & t_K & & t_R \\
 \downarrow & \text{PB} & \downarrow & \text{PO} & \downarrow \\
 L' & \longleftarrow l' & K' & \longrightarrow r' & R
 \end{array}$$

“types”:



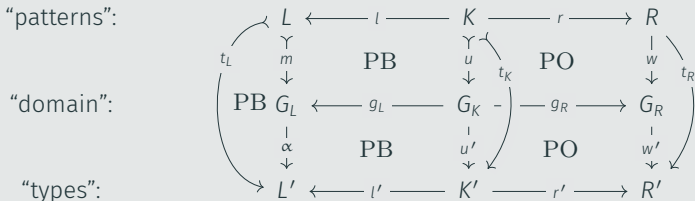
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A step  $G_L \Rightarrow G_R$  is given by:



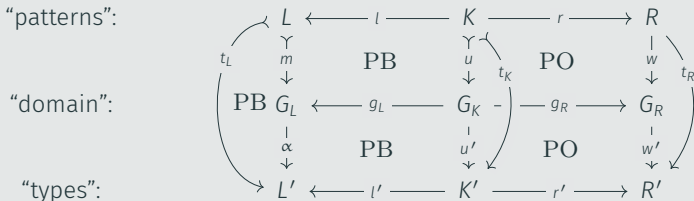
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Allows application conditions, deleting, cloning, adding, merging, ...

# Not All Categories Are Created Equal

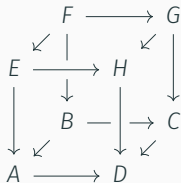
For arbitrary categories:

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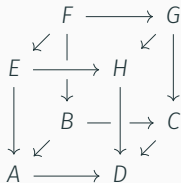
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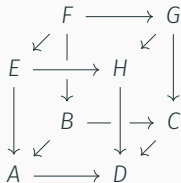


- There may or may not be auxiliary objects and constructions for rewriting or analysis.

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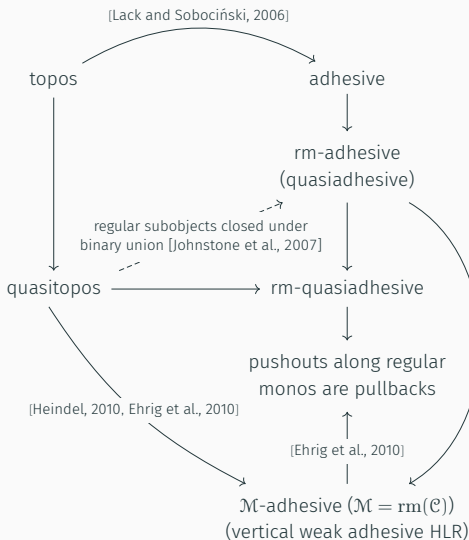
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- There may or may not be auxiliary objects and constructions for rewriting or analysis.

For this reason, there is a **taxonomy** of classes of categories in the literature.

# Taxonomy



Category (underlying data type)	quasi-topos adhesive	quasi-adhesive adhesive HLR	hor. weak. adh. HLR	vert. weak. adh. HLR
<b>Set</b> (sets)	✓	✓	✓	✓
<b>Graph</b> (directed multigraphs)	✓	✓	✓	✓
<b>HyperGraph</b> (directed ordered hypergraphs)	✓	✓	✓	✓
<b>Sig</b> (algebraic signatures)	✓	✓	✓	✓
$\hat{S}$ (presheaves on category $S$ )	✓	✓	✓	✓
$\mathbf{T}_\Sigma$ (term graphs over a signature $\Sigma$ )	?	✓	✓	✓
<b>TripleGraph</b> (functor category $[S_3, \mathbf{Graph}]$ )	?	✓	✓	✓

[Behr et al., 2022, Table 2]

## Recent Subsumption Result

### Definition (Modeling)

A graph rewriting framework  $\mathcal{F}$  is modeled by  $\mathcal{G}$ , denoted  $\mathcal{F} \prec \mathcal{G}$ , if

$$\forall \rho \in \text{rules}(\mathcal{F}). \exists \sigma \in \text{rules}(\mathcal{G}). \Rightarrow_{\rho}^{\mathcal{F}} = \Rightarrow_{\sigma}^{\mathcal{G}}.$$





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### Theorem ([Overbeek et al., 2022])

In any *quasitopos*, using regular monic matches  $m$ :

$$\begin{array}{ccccc}
 & & [Corradini et al., 2015] & & \\
 & & \swarrow & \swarrow & \swarrow \\
 \text{SqPO} & \prec & \text{AGREE} & \prec & \text{PBPO}^+ & \prec & \text{DPO} \\
 & & & & \downarrow & & \\
 & & & & \text{PBPO} & & 
 \end{array}$$

□

Preprint under review, available on arXiv:

Overbeek, R., Endrullis, J., and Rosset, A. (2022). Graph rewriting and relabeling with PBPO<sup>+</sup>: A unifying theory for quasitoposes.

CoRR, abs/2203.01032

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**Natural setting** for assumptions: finitary rm-adhesive quasitoposes.

Includes the category of finite directed multigraphs.

Because of our subsumption result, this yields also a termination method for SqPO, DPO, AGREE, and PBPO rules in this setting.

# Termination by Tiling

Set of weighted tiles  $(\mathbb{T}, w)$ :

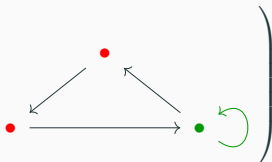
$$\left\{ \left( \bullet, 1 \right), \left( \begin{array}{c} \bullet \\ \swarrow \quad \nwarrow \\ \bullet \quad \longrightarrow \quad \bullet \end{array}, 4 \right), \left( \begin{array}{c} \curvearrowright \\ \bullet \end{array}, 3 \right) \right\}$$

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**Weight of a graph** is the heaviest injective, non-overlapping tiling possible:

$$w \left( \begin{array}{c} \bullet \\ \swarrow \quad \nwarrow \\ \bullet \quad \longrightarrow \quad \bullet \end{array} \right) = 5$$


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## Proposition (Termination by Tiling)

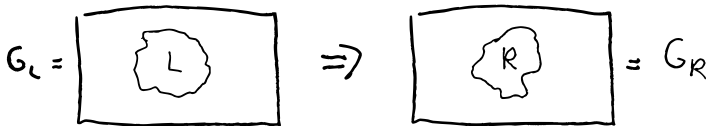
A rewrite system  $R$  is terminating if  $\exists (\mathbb{T}, w)$  such that

$\forall \text{ steps } G_L \Rightarrow_R G_R. \quad w(G_L) > w(G_R).$

□



# Intuitive Idea

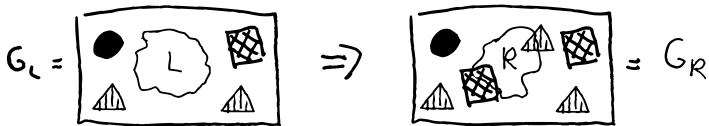


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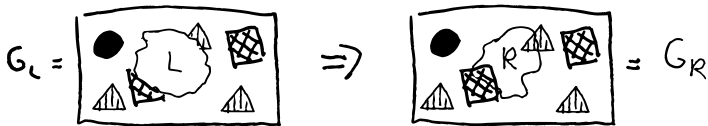
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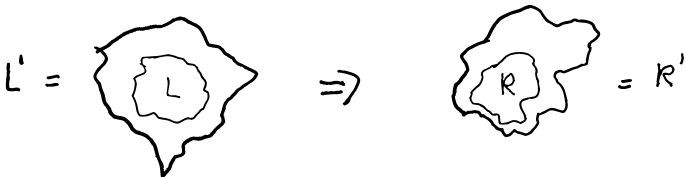
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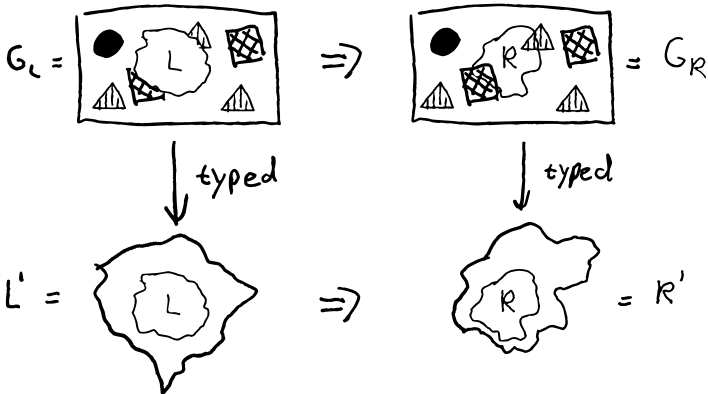


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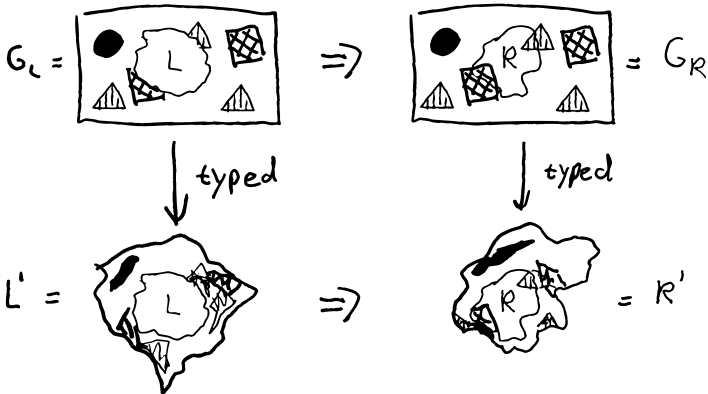
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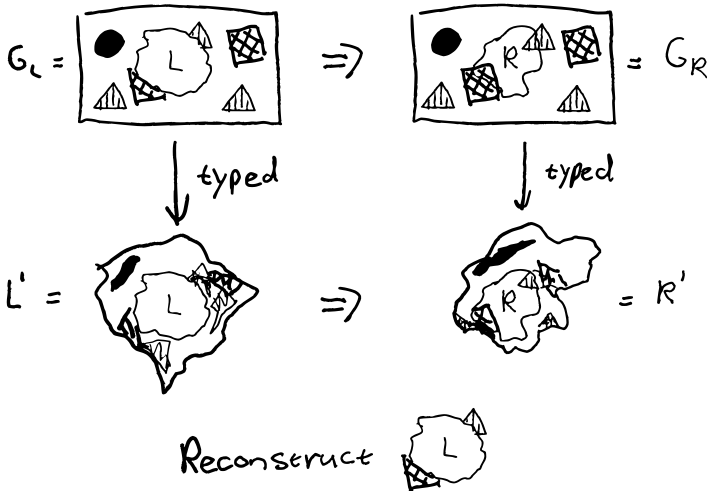
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## Theorem

*Given a rule  $\rho$ , and some assumptions on the category and the rule.*

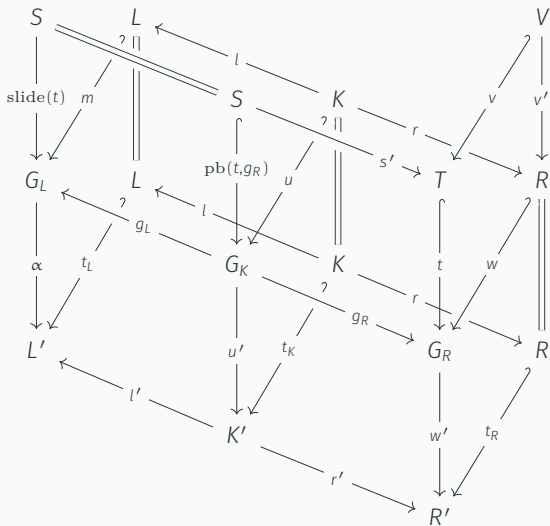
*Suppose that for **every** tiling of  $R$  (where tiles may stick out into the environment typing and become deformed), the reconstruction of  $L$  + the transferred fragments glued around it admits a heavier tiling.*

*Then  $\rho$  is terminating.*



This method must be further relaxed!

## Abstract Picture



## Related Work

Main point of comparison:

Bruggink, H. J. S., König, B., Nolte, D., and [H. Zantema](#) (2015). Proving termination of graph transformation systems using weighted type graphs over semirings.

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More related work to be investigated!

# Conclusion

Termination by Tiling for PBPO<sup>+</sup>: proofs are there, but article in draft phase

Todos:

- relax tiling constraints
- iron out some details
- evaluate strength
- compare with related work

Thank you!