Inclusion Exclusion revisited

Vincent van Oostrom[1]

https://people.bath.ac.uk/vvo21/

[1]Supported by EPSRC Project EP/R029121/1 Typed lambda-calculi with sharing and unsharing.

inclusion / exclusion principle

1st abstraction: IE for multisets

2nd abstraction: IE for commutative residual algebras

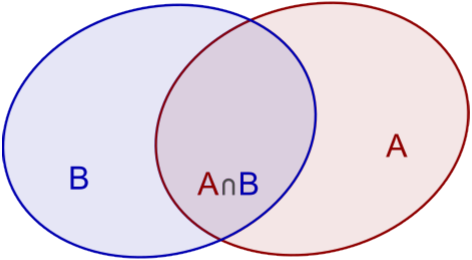
embedding

1st embedding: CRAs in CRAs with composition

2nd embedding: CRACs in commutative ℓ-groups

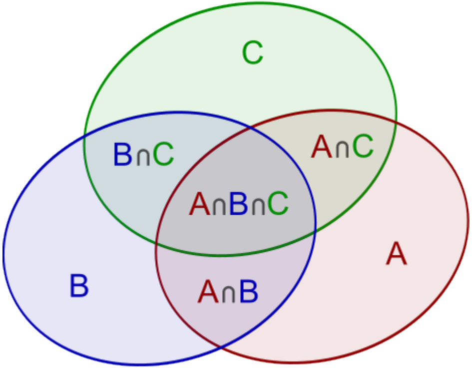
conclusions

Inclusion / exclusion principle for 2 sets (IE_2)



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Inclusion / exclusion principle for 3 sets (IE_3)



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

(picture from Wikipedia)

Inclusion / exclusion principle (IE)

Theorem (Inclusion / exclusion (de Moivre, da Silva, Sylvester C17/18th))

for finite family $A_I := (A_i)_{i \in I}$ *of finite sets*

$$\left|\bigcup A_I\right| = \sum_{\emptyset \subset J \subseteq I} (-1)^{|J| \dot{-} 1} \cdot \left|\bigcap A_J\right|$$

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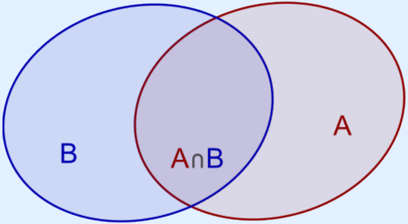
Example

for $I := \{1, 2, 3\}$, $A_1 := \{a, b\}$, $A_2 := \{b, c\}$, and $A_3 := \{c, a\}$

$$|\{a, b, c\}| = 3 = |\{a, b\}| + |\{b, c\}| + |\{c, a\}| - |\{b\}| - |\{c\}| - |\{a\}| + |\emptyset|$$

$\left|\bigcup A_I\right| = \sum_{\emptyset\subset J\subseteq I} (-1)^{|J| \dot{-} 1} \cdot \left|\bigcap A_J\right|$ by double counting

Standard double counting proof.



$|A \cup B| = |A| + |B| - |A \cap B|$ for $x \in A \cap B$?

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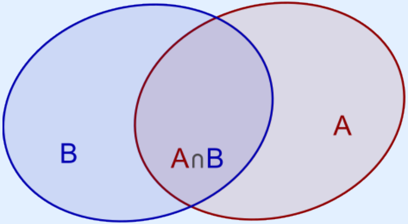

$$1 = 2 - 1 \quad \text{for } x \in {\color{red}A} \cap {\color{blue}B}$$

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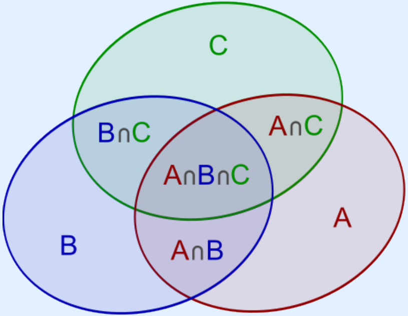

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$ for $x \in A \cap B \cap C$?

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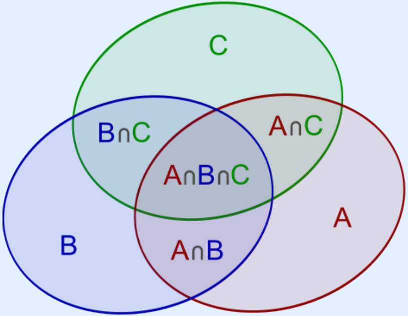

$1 = 3 - 3 + 1 \quad \text{for } x \in A \cap B \cap C$

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count for each individual $x \in \bigcup A_I$ depending on $\#(x) := |\{i \mid x \in A_i\}|$:

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\begin{array}{rcll}
1 & = & 1 & \text{if } \#(x) = 1 \\
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1 & = & 4 - 6 + 4 - 1 & \text{if } \#(x) = 4 \\
1 & = & \dots \quad ? \quad \dots & \text{if } \#(x) = n
\end{array}
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1 & = & \sum_{1\leq j\leq n} (-1)^{j \dot{-} 1} \binom{n}{j} & \text{if } \#(x) = n
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by double counting: $\sum_{0 \leq j \leq n} (-1)^j \binom{n}{j} \Leftarrow (1-1)^n \Rightarrow 0$ ('critical peak') $\square$

Identities analogous to IE_2 (for $|I| = 2$)?

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where $\div$ is truncated division; $x \div y = x / \min(x, y)$

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IE principles for all of these? Yes, and more, and even for partial operations

Prove them uniformly from simple axioms? Yes, by abstracting inductive proof

$\left|\bigcup A_I\right| = \sum_{\emptyset \subset J \subseteq I} (-1)^{|J| \dot{-} 1} \cdot \left|\bigcap A_J\right|$ by induction on #sets $|I|$

Step case $I \cup \{k\}$ of standard proof by induction.

$$\left|\bigcup A_{I \cup \{k\}}\right|$$

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$$\left|\bigcup A_{I \cup \{k\}}\right| \quad =^{\mathrm{IE}_2, \cup\mathrm{semi}\ell} \quad \left|\bigcup A_I\right| + |A_k| - \left|\left(\bigcup A_I\right) \cap A_k\right|$$

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&=^{2\times\mathrm{IH}} \left(\sum_{\emptyset\subset J\subseteq I}(-1)^{|J|\dot{-}1}\cdot\left|\bigcap A_J\right|\right) + |A_k| - \\
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&\quad \sum_{\{k\} \subset J \subseteq I \cup \{k\}} (-1)^{|J| \dot{-} 1} \cdot \left|\bigcap A_J\right|
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\end{aligned}$$

□

1st abstraction: IE for multisets

Example

for $I := \{1, 2, 3\}$, $M_1 := [a, b]$, $M_2 := [b, c]$, and $M_3 := [c, a]$

$$[a, b, c] = [a, b] \uplus [b, c] \uplus [c, a] - [b] - [c] - [a] \uplus \not{[\,]}$$

1^{st} abstraction: IE for multisets

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Idea: multiplicities built in so no need for taking cardinalities by $|\,|$

$\bigcup / \bigcap \mapsto$ multiset union $\cup$ (maximum) / intersection $\cap$ (minimum)

$\sum / - \mapsto$ multiset sum $\uplus$ (addition) / difference $-$ (subtraction)

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caveat: multisets not cgroup; multiplicities nonnegative; cf. $1 = 4 \dot{-} 6 + 4 \dot{-} 1$??

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rearrange to only need cmonoid; $1 = 4 \dot{-} 6 + 4 \dot{-} 1$ into $1 = (4 + 4) \dot{-} (6 + 1)$

1^{st} abstraction: IE for multisets

IE by rearranging odd (positive) and even (negative) summands in IE

$$\left|\bigcup A_I\right| = \sum_{\emptyset \subset J \subseteq I} (-1)^{|J| \dot{-} 1} \cdot \left|\bigcap A_J\right|$$

$$=^{\text{cgroup}} \left(\sum_{\emptyset \subset J \subseteq_{\text{o}} I} \left|\bigcap A_J\right|\right) \dot{-} \left(\sum_{\emptyset \subset J \subseteq_{\text{e}} I} \left|\bigcap A_J\right|\right) \quad \text{since O} \geq \text{E}$$

1^{st} abstraction: IE for multisets

Theorem (IE for finite family of finite multisets / sets)

$$\bigcup M_I = \left(\biguplus_{\emptyset \subset J \underset{o}{\subseteq} I} \bigcap M_J\right) - \left(\biguplus_{\emptyset \subset J \underset{e}{\subseteq} I} \bigcap M_J\right)$$

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Proof.

for multisets: as before (but without $|\,|$ clutter) by induction on $|I|$ also proving O $\underline{\uplus}$ E, again using the algebraic structure laws (only cmonoid; not cgroup)

1st abstraction: IE for multisets

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Proof.

for sets: view as multisets, define $|M| := \sum_x M(x)$, use $|M \uplus N| = |M| + |N|$ and O $\underline{\supseteq}$ E and

$$\left|\bigcup A_I\right| = \left|\left(\biguplus_{\emptyset \subset J \underset{o}{\subseteq} I} \bigcap A_J\right) - \left(\biguplus_{\emptyset \subset J \underset{e}{\subseteq} I} \bigcap A_J\right)\right| = \left(\sum_{\emptyset \subset J \underset{o}{\subseteq} I} \left|\bigcap A_J\right|\right) \dot{-} \left(\sum_{\emptyset \subset J \underset{e}{\subseteq} I} \left|\bigcap A_J\right|\right) \quad \square$$

2nd abstraction: IE for commutative residual algebras

Definition (Commutative Residual Algebra $\langle A, 1, /\rangle$)

$$a/1 = a \tag{1}$$

$$a/a = 1 \tag{2}$$

$$1/a = 1 \tag{3}$$

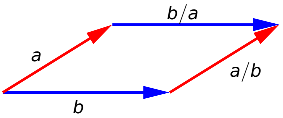
$$(a/b)/(c/b) = (a/c)/(b/c) \tag{4}$$

$$(a/b)/a = 1 \tag{5}$$

$$a/(a/b) = b/(b/a) \tag{6}$$

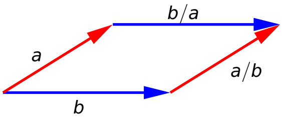
residuation: read a/b as a after b

2^{nd} abstraction: IE for commutative residual algebras



Skolemisation of diamond: $\forall$ peak $a, b.\exists$ valley b', a'

2^{nd} abstraction: IE for commutative residual algebras



Skolemisation of diamond: $\forall$ peak a, b. valley $b/a, a/b$

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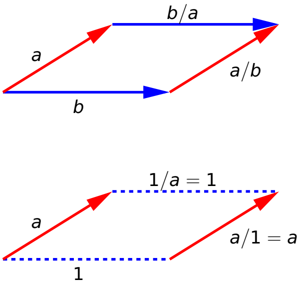
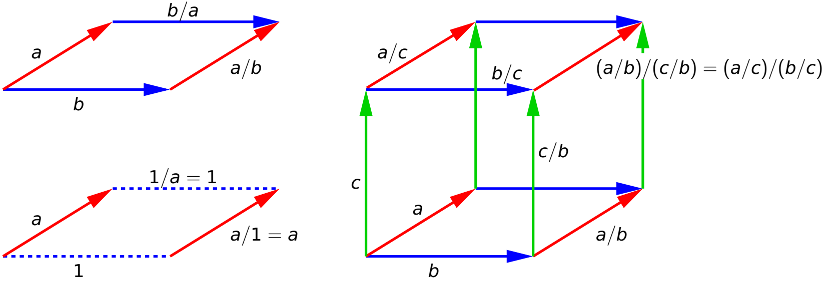
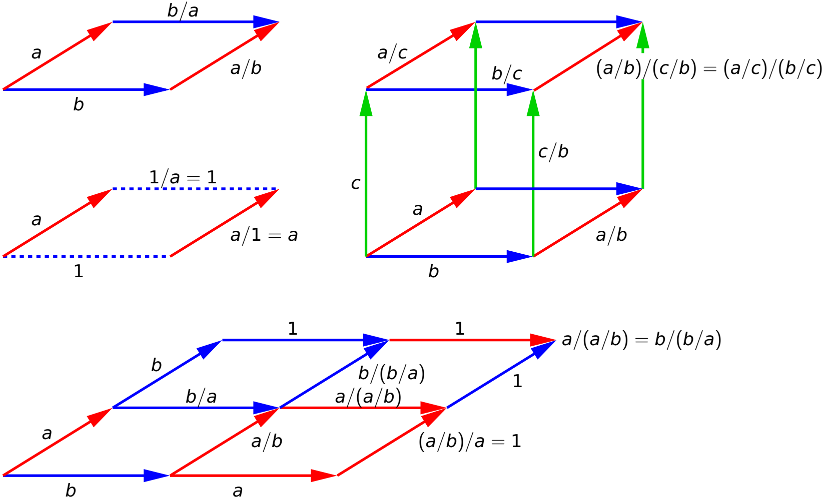


2^{nd} abstraction: IE for commutative residual algebras



laws (1)–(4): residual system (Lévy,Stark,Terese)

2^{nd} abstraction: IE for commutative residual algebras



2^{nd} abstraction: IE for commutative residual algebras

Definition (Commutative Residual Algebra $\langle A, 1, /\rangle$)

$$a/1 = a \tag{1}$$
$$(a/b)/(c/b) = (a/c)/(b/c) \tag{4}$$
$$(a/b)/a = 1 \tag{5}$$
$$a/(a/b) = b/(b/a) \tag{6}$$

Lemma (Some CRA laws)

$$\begin{array}{rclcrcl} a/a & = & 1 & \quad & (a/b)/c & = & (a/c)/b \\ 1/a & = & 1 & \quad & (a/b)/(b/a) & = & a/b \end{array}$$

commutative BCK algebra with relative cancellation (Dvurečenskij & Graziano)

2nd abstraction: IE for commutative residual algebras

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Example (Some CRAs)

$\langle \mathbb{N}, 0, \dot{-} \rangle$, $\langle \text{Pos}, 1, \cdot/ \rangle$, $\langle \text{Mst}(A), [\,], - \rangle$, $\langle \mathbb{R}_{\geq 0}, 0, \dot{-} \rangle$, $\langle \mathbb{R}_{\geq 1}, 1, \div \rangle$, $\ldots$, all mentioned and more: $\langle \{0,1\}, 0, \dot{-} \rangle$, $\langle \text{Set}(A), \emptyset, - \rangle$, $\ldots$, sub-CRAs by downward-closing

CRA derived operations

Definition (Derived operations $\leqslant$, $\wedge$, $\cdot$, $\vee$ for CRA $\langle A, 1, / \rangle$)

$$
\begin{aligned}
a \leqslant b &:= a/b = 1 \\
a \wedge b &:= a/(a/b) \\
a \cdot b &:= c \qquad \text{if } a/c = 1 \text{ and } c/a = b \qquad \text{(partial)} \\
a \vee b &:= a \cdot (b/a) \qquad \text{(partial)}
\end{aligned}
$$

CRA derived operations

Example

	CRA	$\mathbb{N}$	$\mathbb{R}_{\geq 0}$	Mst(A)	Set(A)	Pos
unit	1	0	0	$[\![\,]\!]$	$\emptyset$	1
residual	$/$	$\dot{-}$	$\dot{-}$	$-$	$-$	$\cdot/$
natural order	$\leqslant$	$\leq$	$\leq$	$\subseteq_{+}$	$\subseteq$	$\mid$
total order?		✓	✓	✗	✗	✗
well-founded?		✓	✗	✓ (fin)	✓ (fin)	✓
meet	$\wedge$	min	min	$\cap$	$\cap$	gcd
product	$\cdot$	$+$	$+$	$\uplus$	$\cup$ (if $\downarrow$)	$\cdot$
join	$\vee$	max	max	$\cup$	$\cup$	lcm

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Lemma (Satisfaction of algebraic IE laws for CRAs)

- *$\langle A, \leqslant\rangle$ is a partial order; enables proving $a = b$ by inclusions $a \leqslant b$ and $b \leqslant a$*

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- *$\langle A, 1, \cdot\rangle$ is a partial commutative monoid; $a \leqslant b$ iff $a \cdot c = b$ for some c;*

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\end{aligned}
$$

Lemma (Satisfaction of algebraic IE laws for CRAs)

- $\langle A, \leqslant\rangle$ *is a partial ℓattice; $a \vee (a \wedge b) \simeq a$ and $a \wedge (a \vee b) \simeq a$ if $(a \vee b)\downarrow$*
- $\langle A, \wedge\rangle$ *is a meet-semilattice; $a \leqslant b$ iff $a \wedge b = a$; $1 \wedge a = 1$*
- $\langle A, 1, \cdot\rangle$ *is a partial commutative monoid; $a \leqslant b$ iff $a \cdot c = b$ for some c;*
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Lemma (Satisfaction of algebraic IE laws for CRAs)

- $\langle A, \leqslant\rangle$ *is a partial lattice;* $a \vee (a \wedge b) \simeq a$ *and* $a \wedge (a \vee b) \simeq a$ *if* $(a \vee b)\downarrow$
- $\langle A, \wedge, \vee\rangle$ *is a partial distributive lattice;* $(a \vee b) \wedge c \simeq (a \wedge c) \vee (b \wedge c)$ *if* $(a \vee b)\downarrow$
- $\langle A, 1, \cdot\rangle$ *is a partial commutative monoid;* $a \leqslant b$ *iff* $a \cdot c = b$ *for some* c*;*

IE for CRAs

Theorem (Inclusion / exclusion for finite family a_I)

$$O := \left(\prod_{\emptyset \subset J \underset{o}{\subseteq} I} \bigwedge a_J \right) \downarrow \quad \text{and} \quad E := \left(\prod_{\emptyset \subset J \underset{e}{\subseteq} I} \bigwedge a_J \right) \downarrow \implies \bigvee a_I \simeq O/E \quad \text{and} \quad E \leqslant O$$

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Proof.

algebraic version of proof by induction on $|I|$ using IE laws for CRAs and

$$
\begin{array}{rcll}
(b/a) \wedge (c/a) & = & (c/a)/(c/b) & = \quad (b \wedge c)/(a \wedge c) \\
(a \cdot b)/(c \cdot d) & = & (a/c)/(d/b) & \text{if } c \leqslant a, b \leqslant d \text{ and } (a \cdot b)\downarrow, (c \cdot d)\downarrow \\
(a \cdot b) \wedge c & \simeq & (a \wedge c) \cdot (b \wedge (c/a)) & \text{if } (a \cdot b)\downarrow
\end{array}
$$

□

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How general is this?

Versions of IE we know of are instances, e.g. probabilities.
Novel (?) instance, measurable multisets, next up.

2^{nd} abstraction: IE for measurable multisets

Definition

$\mathcal{A}$ algebra if $\mathcal{A} \subseteq \wp(A)$ with $A \in \mathcal{A}$ and closed under union and complement

2^{nd} abstraction: IE for measurable multisets

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formally, $\mathcal{A}$ sub-algebra of the Boolean algebra $\wp(A)$
simple case of algebra in measure theory where closed under countable union

2^{nd} abstraction: IE for measurable multisets

Definition

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Definition (multiset M being $\mathcal{A}$-measurable)

- $M^i \in \mathcal{A}$ for each i, with $M^i := \{a \mid M(a) = i\}$ (set at height i of M)
- $M^{>i} = \emptyset$ for some i, with $M^{>i} := \bigcup_{j>i} M^j = \{a \mid M(a) > i\}$ (least i is height of M)

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set measurable iff it is so viewed as multiset

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Lemma (CRA)

- *sets M^i at height i partition A*

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- *sets M^i at height i partition A*
- *$M^{>0}$ is support of M (need not be finite!); M empty iff height* 0

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Lemma (CRA)

- *sets M^i at height i partition A*
- *$M^{>0}$ is support of M ; M empty iff height 0*
- $\langle \mathrm{Mst}(\mathcal{A}), [\![\,]\!], - \rangle$ *of $\mathcal{A}$-measurable multisets is CRA (closed under $-$)*

2^{nd} abstraction: IE for measurable multisets

Definition

function μ from algebra $\mathcal{A}$ to non-negative reals is measure if

- $\mu(\emptyset) = 0$
- $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A, B \in \mathcal{A}$ and disjoint

extended to measurable multisets by $\mu(M) := \sum_i \mu(M^{>i})$

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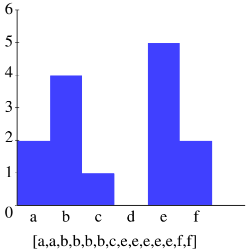

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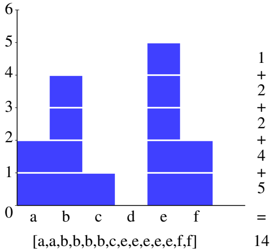

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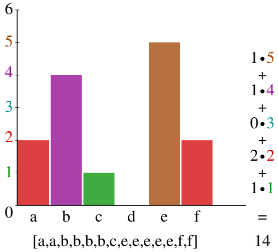
extended to measurable multisets by $\mu(M) := \sum_i \mu(M^{>i}) = \sum_j j \cdot \mu(L^j)$



2^{nd} abstraction: IE for measurable multisets

Theorem (IE for finite family of measurable multisets / sets)

$$\bigcup M_I = \left(\biguplus_{\emptyset \subset J \underset{o}{\subseteq} I} \bigcap M_J\right) - \left(\biguplus_{\emptyset \subset J \underset{e}{\subseteq} I} \bigcap M_J\right)$$

$$\mu(\bigcup A_I) = \left(\sum_{\emptyset \subset J \underset{o}{\subseteq} I} \mu(\bigcap A_J)\right) \dot{-} \left(\sum_{\emptyset \subset J \underset{e}{\subseteq} I} \mu(\bigcap A_J)\right)$$

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Proof.

for measurable multisets: instance of IE for CRAs

□

2nd abstraction: IE for measurable multisets

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Proof.

for sets: view as multisets, use $\mu(M \uplus N) = \mu(M) + \mu(N)$ and $\mathrm{O} \mathrel{\underline{\supset\!\!\!+}} \mathrm{E}$ and

$$\mu(\bigcup A_I) = \mu(\left(\biguplus_{\emptyset \subset J \underset{o}{\subseteq} I} \bigcap A_J\right) - \left(\biguplus_{\emptyset \subset J \underset{e}{\subseteq} I} \bigcap A_J\right)) = \left(\sum_{\emptyset \subseteq J \underset{o}{\subseteq} I} \mu(\bigcap A_J)\right) \dot{-} \left(\sum_{\emptyset \subset J \underset{e}{\subseteq} I} \mu(\bigcap A_J)\right) \qquad \square$$

the IE?

Holler

I like neither the partial · nor the binary / because they are so ugly !

the IE?

Holler

I want my beautiful total composition / product and unary inverse !!

the IE!

Holler

I want my beautiful total composition / product and unary inverse !!

Sooth

bits	:	natural numbers	:	integers
$\langle \mathbb{B}, 0, \dot{-} \rangle$	:	$\langle \mathbb{N}, 0, \dot{-}, + \rangle$	:	$\langle \mathbb{Z}, 0, (-), +, \min, \max \rangle$
CRA	:	CRAC	:	commutative lattice-ordered group
$\langle A, 1, / \rangle$	:	$\langle A, 1, /, \cdot \rangle$	:	$\langle A, 1, {}^{-1}, \cdot, \wedge, \vee \rangle$

where $\mathbb{B} := \{0, 1\}$ and a CRAC is a CRA with composition $\cdot$

From CRAs to CRACs

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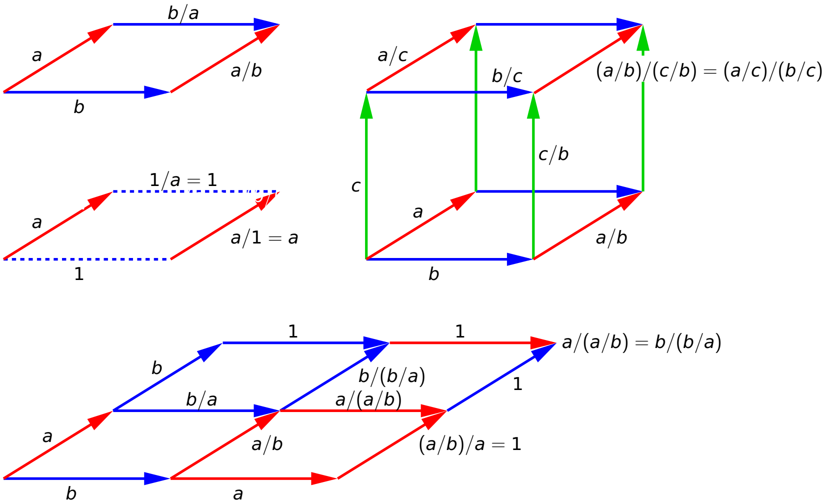
From CRAs to CRACs



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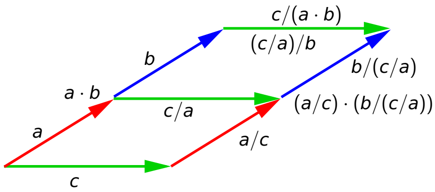

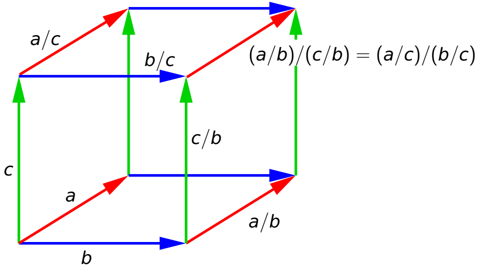

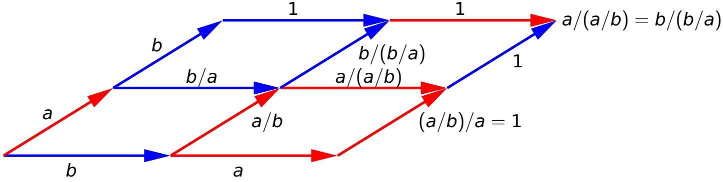

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Composition in CRA vs. CRAC

in CRAs with derived composition $\cdot$ partial versions of (7)–(8) hold:
$c/(a \cdot b) = (c/a)/b$ and $(a \cdot b)/c \simeq (a/c) \cdot (b/(c/a))$ if $(a \cdot b)\downarrow$, and $1 \cdot 1 = 1$

From CRAs to CRACs

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$$1 \cdot 1 = 1 \tag{9}$$

Composition in CRAC vs. CRA

composition $\cdot$ in CRACs satisfies the laws of derived composition in CRAs:
$a/(a \cdot b) = (a/a)/b = 1$ and $(a \cdot b)/a = 1 \cdot b = (1 \cdot b)/((1 \cdot b)/b) = b$.

$\mathcal{C} = \langle A, 1, / \rangle$ embeds in CRAC $\mathcal{C}^+ = \langle \mathrm{Mst}(A)/\equiv, [\!], /\!\!/, \uplus \rangle$

Idea (commutative case of residual system construction)

adjoin compositions of objects freely, modulo AC, respecting residuation

$\mathcal{C} = \langle A, 1, / \rangle$ embeds in CRAC $\mathcal{C}^+ = \langle \mathrm{Mst}(A)/{\equiv}, \not[\,], /\!\!/, \uplus \rangle$

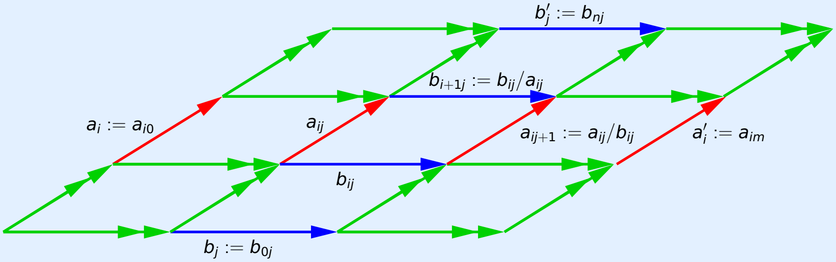
Idea (commutative case of residual system construction)

carrier of $\mathcal{C}^+$: multisets of objects modulo projection equivalence

Residuation in $\mathcal{C}^+$ by tiling with diamonds of $\mathcal{C}$ (Lévy, Stark, Terese)

abbreviate $[a_I]$ to $[a_i \mid i \in I]$. let $I := \{0, \ldots, n \dot{-} 1\}$ and $J := \{0, \ldots, m \dot{-} 1\}$.

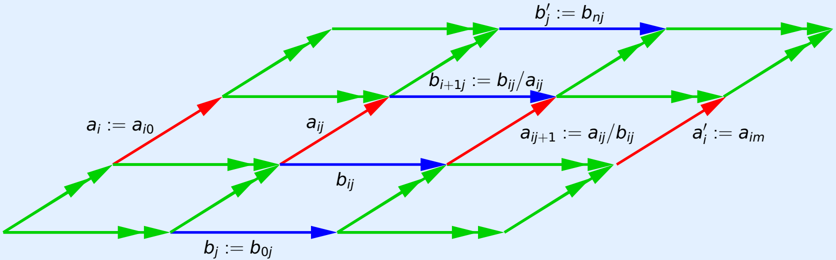
- residual $[a_I] /\!\!/ [b_J]$ of $[a_I]$ (left) after $[a_I]$ (bottom) is $[a'_I]$ (right) by tiling



Residuation in $\mathcal{C}^+$ by tiling with diamonds of $\mathcal{C}$

abbreviate $[a_I]$ to $[a_i \mid i \in I]$. let $I := \{0, \ldots, n \dot{-} 1\}$ and $J := \{0, \ldots, m \dot{-} 1\}$.

- residual $[a_I] /\!\!/ [b_J]$ of $[a_I]$ after $[a_I]$ is $[a'_I]$ by tiling
- projection equivalence $[a_I] \equiv [b_J]$ if $[a'_I] = [\!\!/] = [b'_J]$ ($[a_I] \sqsubseteq [b_J]$ and $[b_J] \sqsubseteq [a_I]$)



$\mathcal{C} = \langle A, 1, / \rangle$ embeds in CRAC $\mathcal{C}^{+} = \langle \mathrm{Mst}(A)/\equiv, [\,]\!\!\!/\,, /\!\!/, \uplus \rangle$

Definition (of components of $\mathcal{C}^{+}$)

- carrier: finite multisets over A modulo projection equivalence $\equiv$ (projection may be computed by rules (7)–(8) eliding units 1 of $\mathcal{C}$)

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- carrier: finite multisets over A modulo projection equivalence $\equiv$
- unit: empty multiset $[\!/\!]$
- residuation $/\!\!/$: by tiling with diamonds of $\mathcal{C}$
- composition: multiset sum $\uplus$
- embedding of $\mathcal{C}$ into $\mathcal{C}^+$: $a \mapsto [a]$

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Example

CRA $\mathcal{C} := \langle \{0, \dots, 9\}, 0, \dot{-} \rangle$ of digits;

- has some compositions, e.g. $7 = 3 + 4$ (since $3 \dot{-} 7 = 0$ and $7 \dot{-} 3 = 4$) but most not, e.g. each of $7 + 6$, $9 + 4$, $4 + 4 + 4$ not defined in $\mathcal{C}$

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- $[7, 6] /\!\!/ [9, 4] \overset{(7)}{=} ([7, 6] /\!\!/ [9]) /\!\!/ [4] \overset{(8)}{=} [7 \dot{-} 9, 6 \dot{-} (9 \dot{-} 7)] /\!\!/ [4] = [0, 4] /\!\!/ [4] = [\!/\!]$
  $[9, 4] /\!\!/ [7, 6] \overset{(7)}{=} ([9, 4] /\!\!/ [7]) /\!\!/ [6] \overset{(8)}{=} [9 \dot{-} 7, 4 \dot{-} (7 \dot{-} 9)] /\!\!/ [6] = [2, 4] /\!\!/ [6] = [\!/\!]$

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- $\mathcal{C}^+$ isomorphic to $\langle \mathbb{N}, 0, \dot{-}, + \rangle$

$\mathcal{C} = \langle A, 1, / \rangle$ embeds in CRAC $\mathcal{C}^+ = \langle \mathrm{Mst}(A)/\!\equiv, \llbracket\!\!/\rrbracket, /\!\!/, \uplus \rangle$

Theorem (Dvurečenskij)

$\mathcal{C}$ *embeds* $([a] \equiv [b] \implies a = b, [1] \equiv \llbracket\!\!/\rrbracket, [a] /\!\!/ [b] \equiv [a/b])$ *downward-closedly (dc)* $(M /\!\!/ [b] \equiv \llbracket\!\!/\rrbracket \implies \exists\, a.M \equiv [a])$ *in* $\mathcal{C}^+$

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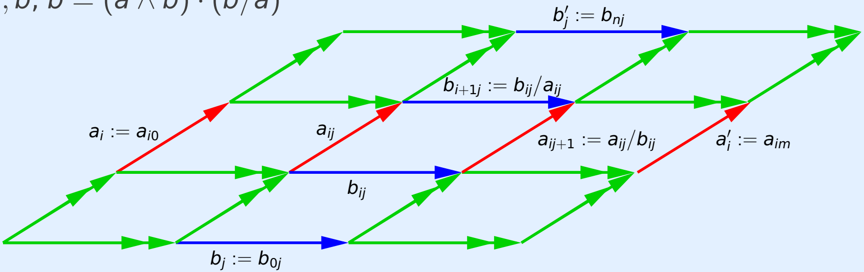
Theorem (Dvurečenskij)

$\mathcal{C}$ embeds downward-closedly in $\mathcal{C}^+$

Proof.

for all a, b, $b = (a \wedge b) \cdot (b/a)$



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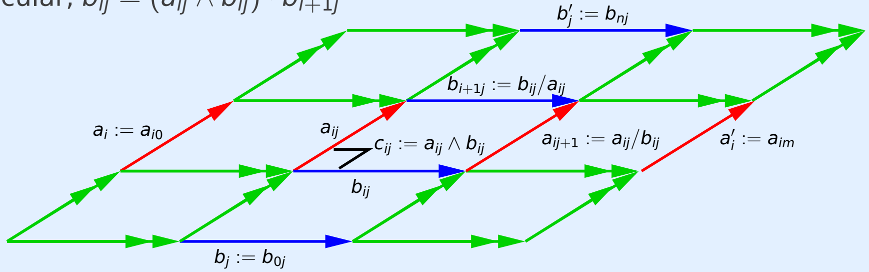
in particular, $b_{ij} = (a_{ij} \wedge b_{ij}) \cdot b_{i+1j}$



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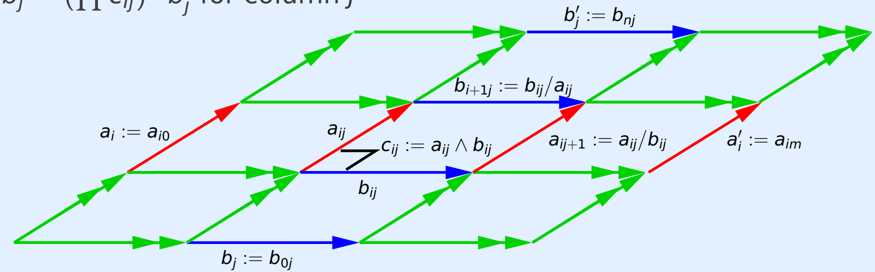
Proof.

hence $b_j = (\prod c_{Ij}) \cdot b'_j$ for column j



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- embedding being trivial ($a = b$ iff $a/b = 1$ and $b/a = 1$), to show downward-closedness assume $M \,/\!/\, [b] \equiv [\,]$ for $M = [a_I]$

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that $\equiv$ is a congruence follows by cubing with (4). $\square$

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Theorem (Dvurečenskij)

$\mathcal{C}$ embeds downward-closedly in $\mathcal{C}^+$

Corollary

for CRA-expressions t, s, universal statement $\forall \vec{a}.t = s$ valid in CRAs iff in CRACs

by downward-closedness also bounded $\exists$s could be allowed

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Corollary

any partial monoid homomorphism f from $\mathcal{C}$ into a monoid $\mathcal{M} := \langle B, \mathbb{1}, \circ \rangle$ factors via embedding and a unique monoid homomorpism h from $\mathcal{C}^+$ to $\mathcal{M}$

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Proof.

let f be such that if $c = a \cdot b$ then $f(c) = a \circ b$. by monoid homomorphism only choice $h : [a_0, \ldots, a_{n \dot{-} 1}] \mapsto f(a_0) \circ \ldots \circ f(a_n)$. that h is a function independent of representative, follows from that if $[a_I] \equiv [b_J]$. then $[a_I] \equiv [c_{IJ}] \equiv [b_J]$ for some matrix c_{IJ} (above Riesz decomposition). conclude by rearranging and assumption from $f(a_0) \circ \ldots \circ f(a_{n \dot{-} 1}) = f(c_{00}) \circ \ldots \circ f(c_{(n \dot{-} 1)(m \dot{-} 1)}) = f(b_0) \circ \ldots \circ f(b_{m \dot{-} 1})$. $\square$

From CRACs to commutative ℓ-groups

Definition (Commutative (abelian) lattice-ordered group)

A commutative ℓ-group is structure $\mathcal{G} := \langle A, 1, ^{-1}, \cdot, \wedge, \vee \rangle$ with $\langle A, \wedge, \vee \rangle$ a lattice, $\langle A, 1, ^{-1}, \cdot \rangle$ a commutative group, where $\cdot$ preserves order $a \leqslant b \implies a \cdot c \leqslant b \cdot c$

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Remark

for any such $\mathcal{G}$, lattice $\langle A, \wedge, \vee \rangle$ is distributive

CRAC $\mathcal{C} = \langle A, 1, /, \cdot \rangle$ embeds in commutative ℓ-group $\widehat{\mathcal{C}}$

Idea (construction of commutative group out of commutative monoid)

adjoin inverses of objects freely, modulo AC, respecting cancellation

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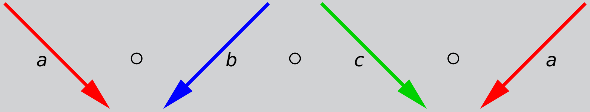
Example

freely compose a, b^{-1} (inverse of b), c, a^{-1} (inverse of a)



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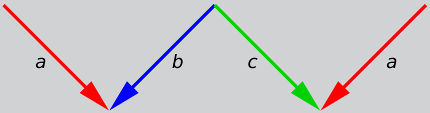
Example

freely compose a, b^{-1} , c, a^{-1} (conversion)



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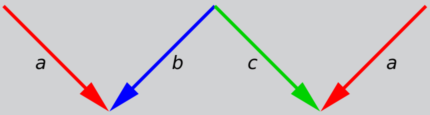
Example

compose a, b^{-1}, c, a^{-1} modulo AC



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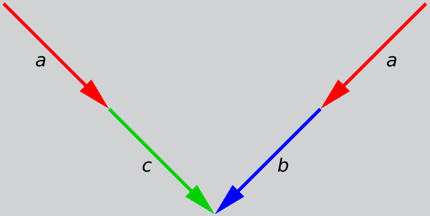
Example

sort into positive–negative (forward–backward) order a, c, b^{-1}, a^{-1}



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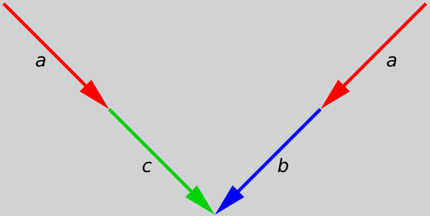
Example

sort into valley $\frac{a \cdot c}{a \cdot b}$; how to respect cancellation?; reordering of a, a^{-1} needed?



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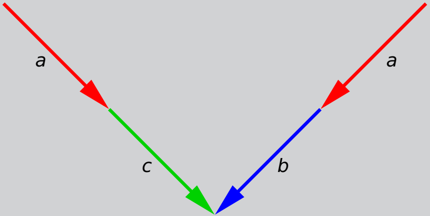
Example

quotient out $\equiv$ as for fractions: $\frac{6}{10} \equiv \frac{21}{35}$ because $6 \cdot 35 = 21 \cdot 10$



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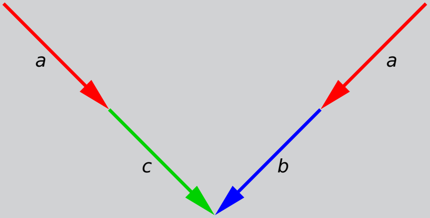
Example

$\frac{a}{b} \equiv \frac{c}{d}$ if $a \cdot d \cdot e = c \cdot b \cdot e$ for some $e \implies$ commutative group (Grothendieck)



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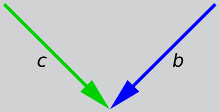
Example

in the present example: $\frac{a \cdot c}{a \cdot b} \equiv \frac{c}{b}$ because $a \cdot c \cdot b = c \cdot a \cdot b$



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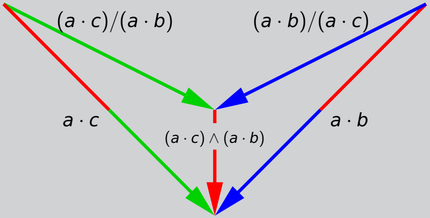
Example

CRACs simpler: have cancellation; cancel $a \wedge b$ from $\frac{a}{b}$, so $\frac{a}{b}$ normalises to $\frac{a/b}{b/a}$



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Definition (of components of $\widehat{\mathcal{C}}$)

- carrier: (formal) fractions $\frac{a}{b}$ with $a, b \in A$ that are normalised: $a \wedge b = 1$

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- join: $\frac{a}{b} \vee \frac{c}{d} := \frac{a \vee c}{b \wedge d}$
- embedding $\widehat{\ }$ of $\mathcal{C}$ in $\widehat{\mathcal{C}}$: $a \mapsto \frac{a}{1}$
  $a/b \mapsto (\widehat{a} \cdot (\widehat{b})^{-1}) \vee 1$, other operations to 'themselves'

CRAC $\mathcal{C}=\langle A, 1, /, \cdot\rangle$ embeds in commutative ℓ-group $\widehat{\mathcal{C}}$

Definition

$\widehat{\mathcal{C}} := \langle\{\frac{a}{b} \mid a \wedge b = 1\}, \frac{1}{1}, (\frac{a}{b})^{-1} := \frac{b}{a}, \frac{a}{b} \cdot \frac{c}{d} := \frac{(a/d)\cdot(c/b)}{(d/a)\cdot(b/c)}, \frac{a}{b} \wedge \frac{c}{d} := \frac{a \wedge c}{b \vee d}, \frac{a}{b} \vee \frac{c}{d} := \frac{a \vee c}{b \wedge d}\rangle$

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Example

$\langle\mathbb{N}, 1, \dot{-}/, \cdot\rangle$ gives (normalised) fractions $\frac{n}{m}$; $\frac{6}{5} \cdot \frac{5}{2} = \frac{3}{1}$, $\frac{6}{5} \wedge \frac{5}{2} = \frac{1}{10}$, and $\frac{6}{5} \vee \frac{5}{2} = \frac{30}{1}$.

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Theorem (Dvurečenskij)

$\mathcal{C}$ embeds in positive cone $\widehat{\mathcal{C}}_{\geq 1}$ (elements $\geq$ unit) of commutative ℓ-group $\widehat{\mathcal{C}}$

Easy using ATP e.g. Prover9

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Corollary

for CRA-expressions t, s, universal statement $\forall\vec{a}.t = s$ is valid in CRACs iff $\forall\vec{a} \in \mathcal{G}_{\geq 1}.\widehat{t} = \widehat{s}$ is in commutative ℓ-groups $\mathcal{G}$, for $\widehat{\ }$ such that $\widehat{r/u} := (\widehat{r} \cdot (\widehat{u})^{-1}) \vee 1$

latter decidable (co-NP; Khisamiev, Weispfenning), so former decidable for CRAs

IE for commutative ℓ-groups $\mathcal{G} := \langle A, 1, ^{-1}, \cdot, \wedge, \vee \rangle$

Example

- $\max(6, 15, 10) = \min(6, 15, 10) + 6 + 15 + 10 \dot{-} \min(6, 15) \dot{-} \min(15, 10) \dot{-} \min(10, 6)$

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$$\bigvee a_I = \prod_{\emptyset \subset J \subseteq I} (\bigwedge a_J)^{(-1)^{|J| \dot{-} 1}}$$

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Proof.

as before by induction on $|I|$ now using commutative ℓ-group laws □

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Remark

alternatively in case all elements in a_I are in positive cone $\mathcal{G}_{\geq 1}$:
rearrange rhs using $\cdot$ associ/commutative, ${}^{-1}$ anti-automorphic as:

$$\left(\prod_{\emptyset \subseteq J \subseteq_{o} I} \bigwedge a_J \right) / \left(\prod_{\emptyset \subset J \subseteq_{e} I} \bigwedge a_J \right)$$

and conclude by assumption from IE for CRAs (then also gives $O \geq E$)

Conclusions and questions / projects

commutative case

bits	:	natural numbers	:	integers
$\langle \mathbb{B}, 0, \dot{-} \rangle$	:	$\langle \mathbb{N}, 0, \dot{-}, + \rangle$	:	$\langle \mathbb{Z}, 0, (-), +, \min, \max \rangle$
CRA	:	CRAC	:	commutative ℓ-group
$\langle A, 1, / \rangle$	:	$\langle A, 1, /, \cdot \rangle$	:	$\langle A, 1, {}^{-1}, \cdot, \wedge, \vee \rangle$

Conclusions and questions / projects

non-commutative case?

rewrite system (Newman)	:	category	:	groupoid
multistep / development	:	rewrite sequence	:	valley (Church & Rosser)
simple braids	:	positive braids	:	braids
?	:	?	:	Garside theory (Dehornoy)
residual system	:	RS with composition	:	? (see appendix)
parallel	:	sequential	:	invert

residual system / residual system with composition $\triangleq$
concurrent transition / computation system (Stark)

Further questions / projects

- decide equational theory of CRAs / CRACs / commutative ℓ-groups by TRS?

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  Isabelle / Coq theories of multisets distinct (finite / infinite support; CRAs!)
- conversions are obtained by closing symmetrically then transitively
  the same for CRA $\mathcal{C}$: first $\widehat{\mathcal{C}}$ (not a partial group; what?) then $(\widehat{\mathcal{C}})^+$?

Some opinions

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   $a \cdot b \cdot a^{-1} \cdot b^{-1}$ commutator measures non-commutativity of peak (a, b)
   $(a/b) \cdot (b/a)$ measures metric distance of peak (a, b) in CRACs
   $\phi + \psi - \omega - \chi$ measures balance of peak (ϕ, χ) with valley (ψ, ω) (Newman)
   valley $(\psi/\phi, \phi/\psi)$ witnesses orthogonality / lub / lcm / pushout of peak (ϕ, ψ)

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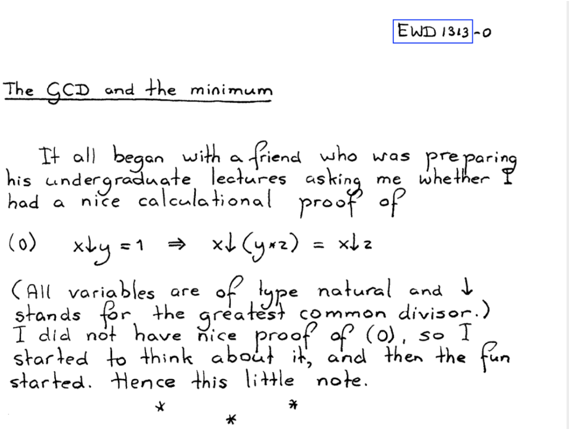
10. the rewriting world will miss the contributions by Hans

CRA problems in disguise: EWD 1313



EWD 1313-0

The GCD and the minimum

It all began with a friend who was preparing his undergraduate lectures asking me whether I had a nice calculational proof of

(0) $x \downarrow y = 1 \;\Rightarrow\; x \downarrow (y * z) = x \downarrow z$

(All variables are of type natural and $\downarrow$ stands for the greatest common divisor.) I did not have nice proof of (0), so I started to think about it, and then the fun started. Hence this little note.

* * *

Edsger W. Dijkstra Archive, EWD 1313, Austin, 27 November 2001

Calculational proof of EWD1313 in CRAs

Lemma

$a \wedge b = 1 \implies a \wedge d = a \wedge c$ *if* $d := b \cdot c$ *is defined, i.e.* $d/b = c$ *and* $b/d = 1$

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Proof.

$$
\begin{aligned}
a \wedge d &\overset{\text{meet}}{=} a/(a/d) \\
&\overset{(1)}{=} a/((a/d)/1) \\
&\overset{\text{hyp}}{=} a/((a/d)/(b/d)) \\
&\overset{(4)}{=} a/((a/b)/(d/b)) \\
&\overset{\text{hyp}}{=} a/(a/(d/b)) \\
&\overset{\text{hyp,meet}}{=} a \wedge c \quad \square
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- *for positive numbers,* $\gcd(n, m) = 1 \implies \gcd(n, m \cdot k) = \gcd(n, k)$

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Example

similar example features in Mechanical Mathematicians (Bentkamp &al.)

$$(\mathrm{gcd}(n,m) = 1 \text{ and } \ell \mid n \cdot m \text{ and } n' = \mathrm{gcd}(\ell, n) \text{ and } m' = \mathrm{gcd}(\ell, m)) \implies$$

$$(n' \cdot m' \mid \ell \text{ and } \ell \mid n' \cdot m')$$

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$$(n' \cdot m' \mid \ell \text{ and } \ell \mid n' \cdot m')$$

is a consequence of a provable CRA statement:

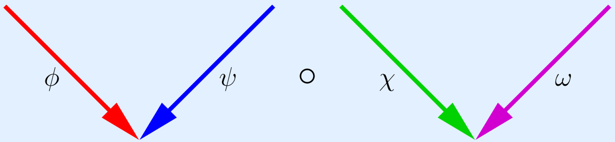
$$\text{if } a \wedge b = 1 \text{ and } (a \cdot b){\downarrow} \text{ and } d \leqslant a \cdot b, \text{ then } (d \wedge a) \cdot (d \wedge b) \simeq d$$

Residual algebra?

Residual systems?

Embedding residual systems with composition in groupoids?

RSC embeds in (typed) involutive monoid of valleys with composition $\circ$:



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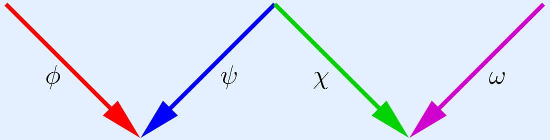

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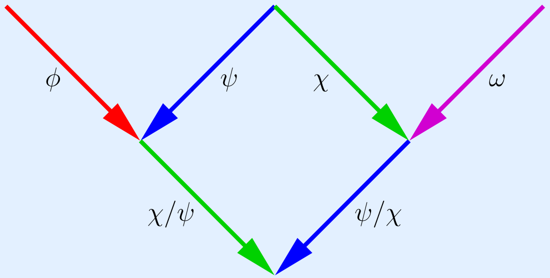

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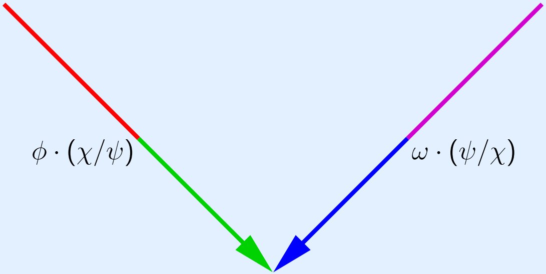

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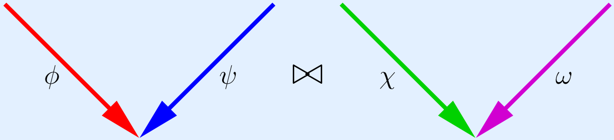
induces groupoid by quotienting out $\bowtie$:



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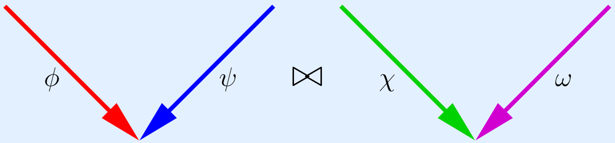
$(\phi, \psi) \bowtie (\chi, \omega)$ if some valley makes both peaks (ϕ, χ) and (ψ, ω) commute:



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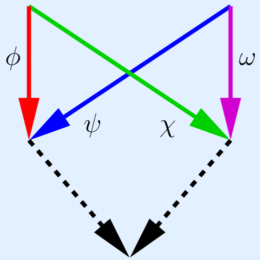

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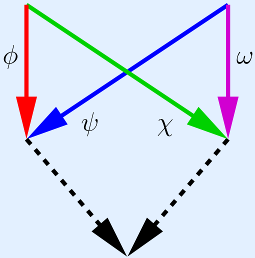

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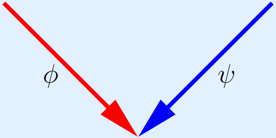
Normalisation trick for braids: reversing (Dehornoy et al.)

if steps can be reversed, normalise via join in reverse system:



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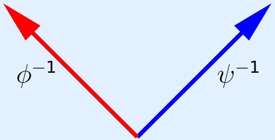

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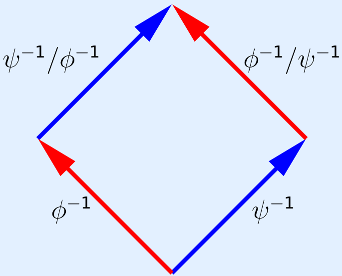

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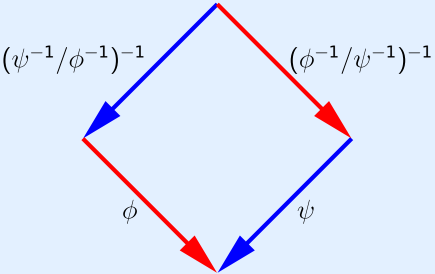

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